SURA'S

Business Mathematics and Statistics

12th Std

Based on the Updated New Textbook

FREE Practice Workbook with Practice Model Question Papers

Salient Features

- Exhaustive Additional Questions & Answers in all chapters.
- **Model Question Papers 1 to 6 (PTA) :** Questions are incorporated in the appropriate sections.
- Govt. Model Question Paper-2019 [Govt. MQP-2019], Quarterly Exam - 2019 [QY-2019], Half yearly Exam - 2019 [HY-2019], Public Exam March- 2020 [March-2020] and Govt. Supplementary Exam September - 2020 & August - 2021 [Sep -2020 & Aug. -2021] are incorporated in the appropriate sections.
- Govt. Supplementary Examination August 2021 Question Paper is given with answers.

SURA PUBLICATIONS
Chennai

2022-23 Edition

ISBN : 978-93-92559-09-9

Code No : SG 325

Authors :

Mr. A.Ganesa Moorthy M.Sc., M.Ed., M.phil.
Chennai

Edited by :
Mr.S. Sathish M.Sc., M.Phil.

Reviewed by :
Mr.S. Niranjan, B.Tech., M.B.A(IIM)
Chennai

Head Office:

1620, 'J' Block, 16th Main Road,
Anna Nagar, **Chennai - 600 040.**
Phones: 044-4862 9977, 044-486 27755.
Mob : 81242 01000 / 81243 01000
e-mail : orders @surabooks.com
website : www.surabooks.com

29-12-2021

PREFACE

The woods are lovely, dark and deep.
But I have promises to keep, and
miles to go before I sleep

\- Robert Frost

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers, and dear Students.

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **SURA'S Business Mathematics and Statistics** for +2 Standard. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises, additional problems and 1 marks as per new model in addition to precise answers for exercise problems.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

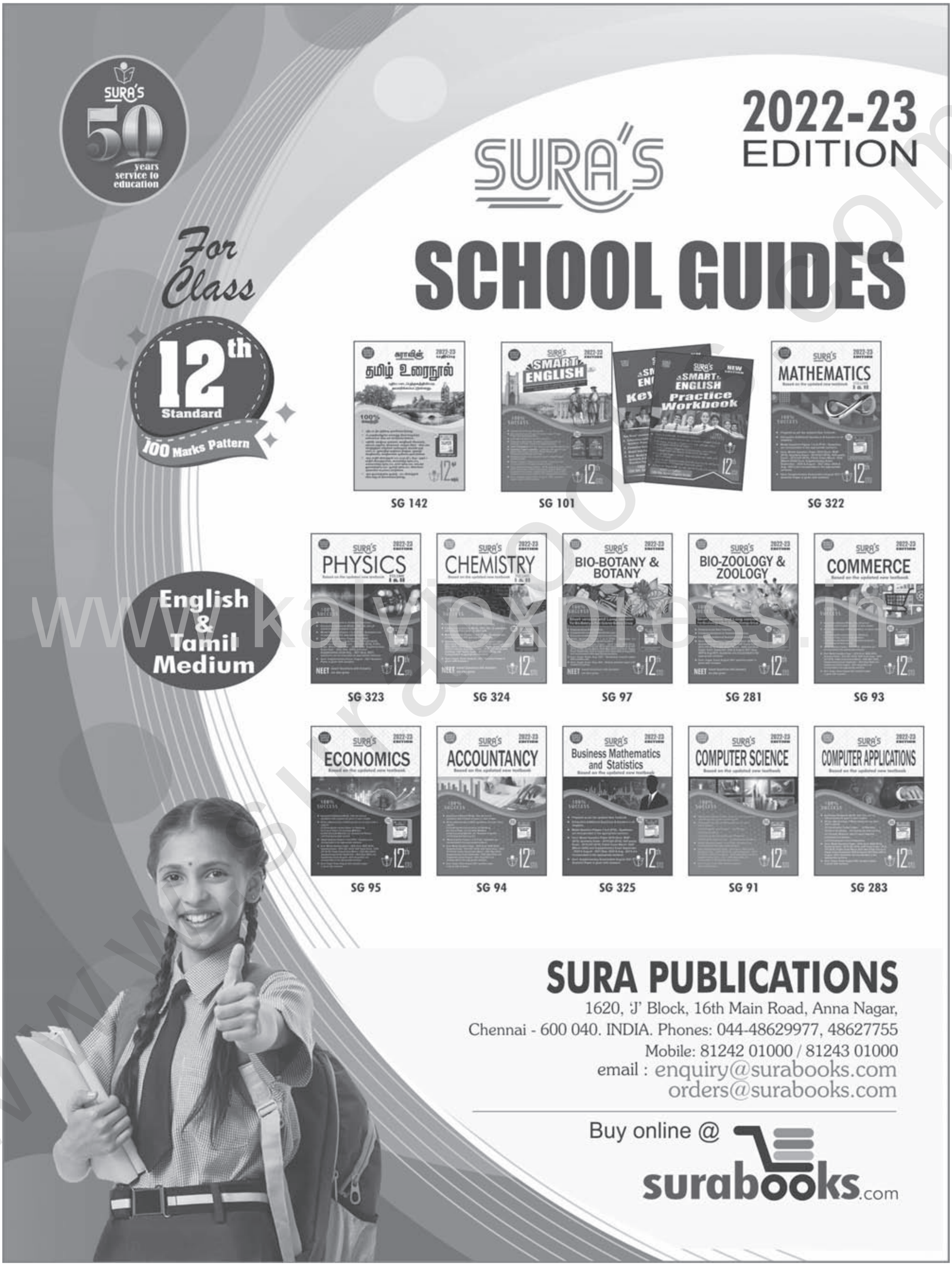
I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

Subash Raj, B.E., M.S.
- Publisher
Sura Publications

All the Best

CONTENTS


SURA'S 50 years service to education
SURA'S
2022-23 EDITION
SCHOOL GUIDES
For Class
12th Standard
100 Marks Pattern
English & Tamil Medium
SG 142
SG 101
SG 322
PHYSICS
CHEMISTRY
BIO-BOTANY & BOTANY
BIO-ZOOLOGY & ZOOLOGY
COMMERCE
SG 323
SG 324
SG 97
SG 281
SG 93
ECONOMICS
ACCOUNTANCY
Business Mathematics and Statistics
COMPUTER SCIENCE
COMPUTER APPLICATIONS
SG 95
SG 94
SG 325
SG 91
SG 283
SURA PUBLICATIONS
1620, 'J' Block, 16th Main Road, Anna Nagar,
Chennai - 600 040. INDIA. Phones: 044-48629977, 48627755
Mobile: 81242 01000 / 81243 01000
email : enquiry@surabooks.com
orders@surabooks.com
Buy online @
surabooks.com

Chapter 1

APPLICATIONS OF MATRICES AND DETERMINANTS

CHAPTER SNAPSHOT

Rank of a matrix :-

The rank of a matrix A is the order of the largest non-zero minor of A and is denoted by ρ (A).

(i) $\rho(A) \geq 0$.

(ii) If A is a matrix of order $m \times n$, then $\rho(A) \leq \min\{m, n\}$.

(iii) Rank of a zero matrix is 0.

(iv) The rank of a non - singular matrix of order $n \times n$ is "n".

Elementary transformations :

(i) Interchange any two rows (or columns)

$$R_i \leftrightarrow R_j \ (C_i \leftrightarrow C_j)$$

(ii) Multiplication of each element of a row (or column) by any non-zero scalar k.

$$R_i \rightarrow k R_i \ (\text{or } C_i \rightarrow k C_i)$$

(iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column).

$$R_i \rightarrow R_i + k R_j \ (\text{or } C_i \rightarrow C_i + k C_j)$$

Equivalent matrices:

Two matrices A and B are said to be equivalent if one is obtained from the another by applying a finite number of elementary transformations.

$A \cong b$

Echelon form :

A matrix A of order $m \times n$ is said to be in echelon form if

(i) Every row of A which has all its entries 0 occurs below every row which has a non - zero entry.

(ii) The number of zeros before the first non - zero element in a row is less than the number of such zeros in the next row.

Transition matrix :

The transition probabilities P_{jk} satisfy $P_{jk} > 0$, $\sum_k P_{jk} = 1$ for all j

FORMULAE TO REMEMBER

1. Linear equations can be written in matrix form AX = B, then the solution is $X = A^{-1}B$, provided $|A| \neq 0$.
2. Consistency of non homogeneous linear equations by rank method.
   - **(i)** If $\rho([A,B]) = \rho(A)$, then the equations are consistent.
   - **(ii)** If $\rho([A,B]) = \rho(A) = n$, where n is the number of variables then the equations are consistent and have unique solution.
   - **(iii)** If $\rho([A,B]) = \rho(A) < n$, then the equations are consistent and have infinitely many solutions.
   - **(iv)** If $\rho([A,B]) \neq \rho(A)$, then the equations are inconsistent and has no solution.
3. Solving non-homogeneous linear equations by Cramer's rule.

If $a_1x + b_1y + c_1z = d_1,$
$a_2x + b_2y + c_2z = d_2,$
$a_3x + b_3y + c_3z = d_3$

Then $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,\ \Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$

$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\quad \Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Then $x = \dfrac{\Delta x}{\Delta},\ y = \dfrac{\Delta y}{\Delta}$ and $z = \dfrac{\Delta z}{\Delta}$

TEXTUAL QUESTIONS

EXERCISE 1.1

1. Find the rank of each of the following matrices.

(i) $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ **(iv)** $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$

(v) $\begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$ **(vi)** $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

(vii) $\begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$ **(viii)** $\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

Sol : (i) Let A $= \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

Order of A is 2 × 2

$\therefore \rho(A) \leq 2$ [Since minimum of (2, 2) is 2]

Consider the second order minor,

$\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = 40 - 42$
$= -2 \neq 0.$

There is a minor of order 2, which is not zero

$\therefore \rho(A) = 2$

(ii) Let A $= \begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$

Order of A is 2 × 2

$\therefore \rho(A) \leq 2$ [Since minimum of (2, 2) is 2]

Consider the second order minor,

$\begin{vmatrix} 1 & -1 \\ 3 & -6 \end{vmatrix} = -6 - (-3)$
$= -6 + 3$
$= -3 \neq 0.$

There is a minor of order 2, which is not zero

$\therefore \rho(A) = 2.$

(iii) Let A $= \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ **[QY-2019]**

Order of A is 2 × 2 [Since minimum of (2,2) is 2]

Consider the second order minor $\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}$
$= 8 - 8$
$= 0.$

Since the second order minor vanishes, $\rho(A) \neq 2$

Consider a first order minor $|1| \neq 0$

There is a minor of order 1, which is not zero

$\therefore \rho(A) = 1.$

(iv) Let A $= \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$

[PTA - 1; Aug.-2021]

The order of A is 3 × 3

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -5 \\ 2 & -1 & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -8 \\ 0 & -3 & -1 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & -3 & -1 \end{pmatrix}$	$R_2 \to R_2 \div 2$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 11 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

This matrix is in echelon from and number of non-zero rows is 3.

$\therefore \rho(A) = 3.$

(v) Let $A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$

The order of A is 3 × 3

$\therefore \rho(A) \le 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$	$R_1 \to R_1 (-1)$
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -4 \\ -2 & 4 & -4 \end{pmatrix}$	$R_2 \to R_2 - 4R_1$
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 + 2R_1$

The matrix is in echelon form and the number of non-zero rows is 2.

$\therefore \rho(A) = 2.$

(vi) Let $A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

The order of A is 3 × 4

$\therefore \rho(A) \le 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 3 & 6 & 3 & -7 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non- zero rows is 2.

$\therefore \rho(A) = 2.$

(vii) $A = \begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$

The order of A is 3 × 4

$\therefore \rho(A) \le 3$ [Since minimum of (3, 4) is 3]

Let us transform the matrix A to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & -5 & -1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 3 & 1 & -5 & -1 \end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 0 & -14 & 16 & -7 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 0 & 0 & 0 & 7 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non-zero matrix is 3.

$\therefore \rho(A) = 3.$

(viii) $A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

The order of A is 3 × 4

$\therefore \rho(A) \leq \text{minimum of } (3, 4) \Rightarrow \rho(A) \leq 3$

Let us transform the matrix A to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ -1 & 2 & 7 & 6 \end{pmatrix}$	$R_2 \to R_2 + 2R_1$
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{pmatrix}$	$R_3 \to R_3 + R_1$
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non- zero rows is 2.

$\therefore \rho(A) = 2.$

2. If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ **and** $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$, **then find the rank of AB and the rank of BA.**

Sol :

Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2-5 & -2+4-1 & 3-6+1 \\ 2+6+20 & -4-12+4 & 6+18-4 \\ 3+4+15 & -6-8+3 & 9+12-3 \end{pmatrix}$$

$$AB = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$$

Matrix (AB)	Elementary Transformation
$AB = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -6 & -2 \\ -12 & 28 & 20 \\ -11 & 22 & 18 \end{pmatrix}$	$C_1 \leftrightarrow C_2$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ -11 & 22 & 18 \end{pmatrix}$	$R_2 \to R_2 + 12R_1$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & -44 & -4 \end{pmatrix}$	$R_3 \to R_3 + 11R_1$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The matrix is in echelon form and the number of non-zero rows is 2.

$\therefore \rho(AB) = 2.$

$$\text{Now, } BA = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4+9 & 1+6-6 & -1-8+9 \\ -2+8-18 & -2-12+12 & 2+16-18 \\ 5+2-3 & 5-3+2 & -5+4-3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 & 0 \\ -12 & -2 & 0 \\ 4 & 4 & -4 \end{pmatrix}$$

Matrix (BA)	Elementary Transformation
$BA = \begin{pmatrix} 6 & 1 & 0 \\ -12 & -2 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 6 & 0 \\ -2 & -12 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	$C_1 \leftrightarrow C_2$
$\sim \begin{pmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	$R_2 \to R_2 + 2R_1$
$\sim \begin{pmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & -20 & -4 \end{pmatrix}$	$R_3 \to R_3 - 4R_1$

The number of non-zero rows is 2.

$\therefore\ \rho(BA) = 2.$

3. Solve the following system of equations by rank method $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x - y - z = 0$

Sol : The given equations are $x + y + z = 9$, $2x + 5y + 7z = 52$, $2x - y - z = 0$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

A X = B

Augmented matrix [AB]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{pmatrix}$	$R_3 \to 3R_3 + R_2$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{pmatrix}$	$\Rightarrow P(A) = 3$

Since augmented matrix $[A, B] \sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & -4 & -20 \end{pmatrix}$ has three non-zero rows, $\rho([A,B]) = 3$.

That is, $\rho(A) = \rho([A,B]) = 3$ = number of unknowns.

So the given system is consistent and has unique solution.

To find the solution, we rewrite the echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 34 \\ -20 \end{pmatrix}$$

$\Rightarrow\ x + y + z = 9$ ... (1)

$3y + 5z = 34$ ...(2)

$-4z = -20$ ...(3)

(3) $\Rightarrow\ -4z = -20$

$z = \frac{-20}{-4} = 5$

(2) $\Rightarrow\ 3y + 5(5) = 34$

$\Rightarrow\ 3y + 25 = 34 \Rightarrow 3y = 34 - 25$

$\Rightarrow\ 3y = 9 \Rightarrow y = \frac{9}{3}$

$y = 3.$

(1) $\Rightarrow\ x + 3 + 5 = 9$

$\Rightarrow\ x + 8 = 9 \Rightarrow x = 9 - 8 \Rightarrow x = 1$

$\therefore\ x = 1,\ y = 3,\ z = 5$ is the unique solution of the given equations.

4. Show that the equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$ are consistent and solve them by rank method.

Sol : Given non-homogeneous equations are

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

A X = B

Augmented matrix $[A, B] = \begin{pmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{pmatrix}$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	
$\sim \begin{pmatrix} 3 & 26 & 2 & 9 \\ 5 & 3 & 7 & 4 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 5 & 3 & 7 & 4 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_1 \to R_1 \div 3$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-121}{3} & \frac{11}{3} & -11 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_2 \to R_2 - 5R_1$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-121}{3} & \frac{11}{3} & -11 \\ 0 & \frac{-176}{3} & \frac{16}{3} & -16 \end{pmatrix}$	$R_3 \to R_3 - 7R_1$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \end{pmatrix}$	$R_2 \to R_2 \div 11$ $R_3 \to R_3 \div 16$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Here $\rho(A) = \rho(A, B) = 2 <$ Number of unknowns.

$\therefore$ The system is consistent with infinitely many solutions let us rewrite the above echelon form into matrix form.

$$\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} \\ 0 & \frac{-11}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$x + \frac{26}{3}y + \frac{2}{3}z = 3 \quad \text{... (1)}$$

$$\frac{-11}{3}y + \frac{1}{3}z = -1 \quad \text{...(2)}$$

let $z = k$; where $k \in R$

$(2) \Rightarrow \frac{-11}{3}y + \frac{k}{3} = -1$

$\Rightarrow \frac{-11}{\cancel{3}}y = -1 - \frac{k}{3} = \frac{-3-k}{\cancel{3}}$

$\Rightarrow -11y = -3 - k$

$\Rightarrow 11y = 3 + k$

$\Rightarrow y = \frac{1}{11}(3 + k)$

Substituting $y = \frac{1}{11}(3 + k)$ and $z = k$ in (1) we get,

$x + \frac{26}{3}\left(\frac{3+k}{11}\right) + \frac{2}{3}k = 3$

$x = -\frac{26}{3}\left(\frac{3+k}{11}\right) - \frac{2k}{3} + 3$

$= \frac{-78-26k}{33} - \frac{2k}{3} + 3 = \frac{-78-26k-22k+99}{33}$

$= \frac{21-48k}{33} = \frac{3(7-16k)}{33}$

$x = \frac{1}{11}(7 - 16k)$

$\therefore$ Solution set is $\{\frac{1}{11}(7-16k), \frac{1}{11}(3+k), k\}\ k \in R$.

Hence, for different values of k, we get infinitely many solutions.

5. Show that the following system of equations have unique solution: $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$ by rank method. [QY-2019]

Sol : Given non-homogeneous equations are

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

A X = B

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

Clearly the last equivalent matrix is in echelon form and it has three non-zero rows.

$\therefore$ ρ(A) = 3 and ρ([A, B]) = 3

$\Rightarrow$ρ(A) = ρ([A, B]) = 3 = Number of unknowns.

$\therefore$ The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x + y + z = 3 \quad$ ... (1)

$y + 2z = 1 \quad$ ... (2)

$2z = 0 \quad$ ... (3)

$(3) \Rightarrow 2z = 0 \Rightarrow z = \frac{0}{2} = 0$

$(2) \Rightarrow y + 2(0) = 1 \Rightarrow y + 0 = 1 \Rightarrow y = 1 - 0 = 1$

$(1) \Rightarrow \quad x + 1 + 0 = 3$

$\Rightarrow \quad x + 1 = 3$

$\Rightarrow \quad x = 3 - 1$

$\Rightarrow \quad x = 2$

$\therefore$ Solution is {2, 1, 0}

6. For what values of the parameter λ, will the following equations fail to have unique solution: 3x– y + λz = 1, 2x + y + z = 2, x + 2y – λz = –1 by rank method.

Sol : Given non-homogeneous equations are

$$3x - y + \lambda z = 1$$
$$2x + y + z = 2$$
$$x + 2y - \lambda z = -1$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 3 & -1 & \lambda \\ 2 & 1 & 1 \\ 1 & 2 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

A X = B

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 3 & -1 & \lambda & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & -\lambda & -1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 3 & -1 & \lambda & 1 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & -7 & 4\lambda & 4 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -1 & \frac{1+2\lambda}{3} & \frac{4}{3} \\ 0 & -1 & \frac{4\lambda}{7} & \frac{4}{7} \end{pmatrix}$	$R_2 \to R_2 \div 3$ $R_3 \to R_3 \div 7$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -1 & \frac{1+2\lambda}{3} & \frac{4}{3} \\ 0 & 0 & \frac{-7-2\lambda}{21} & \frac{-16}{21} \end{pmatrix}$	$R_3 \to R_3 - R_2$

Since

$$\frac{4\lambda}{7} - \frac{1+2\lambda}{3} = \frac{12\lambda - 7 - 14\lambda}{21} = \frac{-7-2\lambda}{21}$$

$$\text{and } \frac{4}{7} - \frac{4}{3} = \frac{12-28}{21} = \frac{-16}{21}$$

Since the system is fail to have unique solution either it can have infinitely many solution or it may be inconsistent.

$\therefore$ This can happen only when $\frac{-7-2\lambda}{21} = 0$.

$$\Rightarrow \quad -7-2\lambda = 0$$

$$\Rightarrow \quad -7 = 2\lambda$$

$$\Rightarrow \quad \lambda = \frac{-7}{2}.$$

7. The price of three commodities X,Y and Z are *x*,*y* and *z* respectively Mr. Anand Purchases 6 units of Z and sells 2 units of X and 3 units of Y. Mr. Amar Purchases a unit of Y and sells 3 units of X and 2units of Z. Mr. Amit Purchases a unit of X and sells 3 units of Y and a unit of Z. In the process they earn ₹ 5,000/-, ₹ 2,000/- and ₹ 5,500/- respectively Find the prices per unit of three commodities by rank method. **[PTA - 5]**

Sol : Given that the price of commodities X, Y and Z are *x*, *y* and *z* respectively.

By the given data,

Transaction	x	y	z	Earning
Mr. Anand	+2	+3	–6	Rs. 5000
Mr. Amar	+3	–1	+2	Rs. 2000
Mr. Amit	–1	+3	+1	Rs. 5500

Here, purchasing is taken as negative symbol and selling is taken as positive symbol.

Thus, the non-homogeneous equations are

$$2x + 3y - 6z = 5000$$
$$3x - y + 2z = 2000$$
$$-x + 3y + z = 5500$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 2 & 3 & -6 \\ 3 & -1 & 2 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5000 \\ 2000 \\ 5500 \end{pmatrix}$$

$$A \quad X = B$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 2 & 3 & -6 & 5,000 \\ 3 & -1 & 2 & 2,000 \\ -1 & 3 & 1 & 5,500 \end{pmatrix}$	
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 3 & -1 & 2 & 2000 \\ 2 & 3 & -6 & 5000 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 0 & 8 & 5 & 18500 \\ 0 & 9 & -4 & 16000 \end{pmatrix}$	$R_2 \to R_2 + 3R_1$ $R_3 \to R_3 + 2R_1$
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 0 & 72 & 45 & 166500 \\ 0 & 72 & -32 & 128000 \end{pmatrix}$	$R_2 \to 9R_2$ $R_3 \to 8R_3$
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 0 & 72 & 45 & 166500 \\ 0 & 0 & -77 & -38500 \end{pmatrix}$	$R_3 \to R_3 - R_2$

$\rho(A) = \rho([A, B]) = 3$ = number of unknowns so the system has unique solution.

$\therefore$ The given system is equivalent to the matrix equation.

$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 72 & 45 \\ 0 & 0 & -77 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5500 \\ 166500 \\ -38500 \end{pmatrix}$$

$$-x + 3y - z = 5500 \quad \text{... (1)}$$
$$72y + 45z = 166500 \quad \text{... (2)}$$
$$-77z = -38500 \quad \text{... (3)}$$

$(3) \Rightarrow \quad -77z = -38500$

$\Rightarrow \quad z = \frac{-38500}{-77} = 500$

$(2) \Rightarrow \quad 72y + 45(500) = 166500$

$\Rightarrow \quad 72y + 22,500 = 166500$

$\Rightarrow \quad 72y = 166500 - 22500$

$\Rightarrow \quad 72y = 144000$

$\Rightarrow \quad y = \frac{144000}{72} \Rightarrow y = 2,000$

$(1) \Rightarrow -x + 3(2000) + 500 = 5500$

$\Rightarrow \quad -x + 6500 = 5500$

$-x = -5500 - 6500$

$-x = -1000$

$x = 1000$

$\therefore$ The prices per unit of the three commodities are ₹1000, ₹ 2000 and ₹ 500.

Chapter 2

INTEGRAL CALCULUS-I

FORMULAE TO REMEMBER

(i) Integration is the reverse process of differentiation

(ii) $\int k\, f(x)\, dx = k \int f(x)\, dx$ where k is a constant.

(iii) $\int [\, f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)\, dx$

(iv) The following are the four principal methods of integration

(i) Integration by decomposition

(ii) Integration by Parts

(iii) Integration by Substitution

(iv) Integration by successive reduction

First fundamental theorem of integral calculus :

If $f(x)$ is a continuous function and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$.

Second fundamental theorem of integral calculus :

$\int_a^b f(x)\,dx = F(b) - F(a)$

$\int_a^b f(x)\,dx$ is a definite constant, whereas $\int_a^x f(t)\,dt$ is a function of the variable x

Indefinite integral :-

An integral function which is expressed without limits, and so containing an arbitrary constant.

Proper definite integral :-

An integral function which has both the limits. a and b are finite.

Improper definite integral :-

An integral function, in which the limits either a or b or both are infinite.

Gamma function :-

For $n > 0$, $\int_0^\infty x^{n-1}e^{-x}\,dx$ and is denoted by $\Gamma(n)$

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

2) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

3) $\int \frac{1}{x} dx = \log|x| + c$

4) $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + c$

5) $\int e^x dx = e^x + c$

6) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

7) $\int a^x dx = \frac{a^x}{\log a} + c,\ a > 0$ and $a \neq 1$

8) $\int a^{mx+n} dx = \frac{1}{m \log a} a^{mx+n} + c,\ a > 0$ and $a \neq 1$

9) $\int \sin x\, dx = -\cos x + c$

10) $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + c$

11) $\int \cos\ dx = \sin x + c$

12) $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + c$

13) $\int \sec^2 x\, dx = \tan x + c$

14) $\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + c$

15) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

16) $\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + c$

17) $\int u\,dv = uv - \int v\,du$ where u and v are two differentiable functions of x [Integration by parts].

18) The code word used in the above formula is
I $\rightarrow$ Inverse trigonometric function
L $\rightarrow$ Logarithmic function
A $\rightarrow$ Algebraic function
T $\rightarrow$ Trigonometric function
E $\rightarrow$ Exponential function

19) Bernoulli's formula :
$\int u\,dv = uv - u' v_1 + u'' v_2 - u''' v_3 + \ldots$
When $u' u'' u''' \ldots$ are the successive derivatives of u and $v_1 v_2 v_3 \ldots$ are the repeated integrals of v.

20) $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$

21) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

22) $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

23) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

24) $\int e^{ax} [a f(x) + f'(x)]\, dx = e^{ax} f(x) + c$

25) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

26) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

27) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}| + c$

28) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}| + c$

29) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2 - a^2}| + c$

30) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c$

Properties of definite integral

1) $\int_a^b f(x)dx = \int_a^b f(t)dt$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

5) $\int_o^a f(x)dx = \int_o^a f(a-x)dx$

6) $\int_{-a}^a f(x)dx = 2\int_o^a f(x)dx$

if $f(x)$ is an even function

7) $\int_{-a}^a f(x)dx = 0$ if $f(x)$ is an odd function

8) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

TEXTUAL QUESTIONS

EXERCISE 2.1

Integrate the following with respect to x

1. $\sqrt{3x+5}$

Sol : $\int \sqrt{3x+5}\, dx = \int (3x+5)^{1/2} dx$

$\left[\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c\right]$

$= \frac{(3x+5)^{\frac{1}{2}+1}}{3\left(\frac{1}{2}+1\right)} + c$

$= \frac{(3x+5)^{3/2}}{3\left(\frac{3}{2}\right)} + c = \frac{(3x+5)^{3/2}}{\frac{9}{2}} + c$

$= \frac{2}{9}(3x+5)^{3/2} + c$

2. $\left(9x^2 - \frac{4}{x^2}\right)^2$

Sol : $\int \left(9x^2 - \frac{4}{x^2}\right)^2 dx$

$= \int \left[(9x^2)^2 - 2(9x^2)\left(\frac{4}{x^2}\right) + \left(\frac{4}{x^2}\right)^2\right] dx$

$[\because (a-b)^2 = a^2 - 2ab + b^2]$

$= \int \left(81x^4 - 72 + \frac{16}{x^4}\right) dx$

$= \int 81x^4\, dx - \int 72dx + \int \frac{16}{x^4}\, dx + c$

$= 81\frac{x^{4+1}}{4+1} - 72x + 16\frac{x^{-4+1}}{-4+1} + c$

$\left[\because \frac{16}{x^4} = 16x^{-4}\right]$

$= 81\frac{x^5}{5} - 72x + 16\frac{x^{-3}}{-3} + c$

$= \frac{81}{5}x^5 - 72x - \frac{16}{3x^3} + c$

3. $(3+x)(2-5x)$ **[Aug. - 2021]**

Sol : $\int (3+x)(2-5x)\, dx$

$= \int (6 - 15x + 2x - 5x^2)\, dx$

$= \int (6 - 13x - 5x^2)\, dx$

$= \int 6dx - \int 13x\, dx - \int 5x^2\, dx$

$= 6x - \frac{13x^2}{2} - \frac{5x^3}{3} + c$

4. $\sqrt{x}\,(x^3 - 2x + 3)$

Sol : $\int \sqrt{x}\,(x^3 - 2x + 3)\,dx$

$= \int x^{\frac{1}{2}}(x^3 - 2x + 3)\,dx$

$= \int \left(x^{3+\frac{1}{2}} - 2x^{1+\frac{1}{2}} + 3x^{\frac{1}{2}}\right)dx$

$= \int x^{\frac{7}{2}}dx - \int 2x^{\frac{3}{2}}dx + \int 3x^{\frac{1}{2}}dx$

$= \dfrac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} - \dfrac{2x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \dfrac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \dfrac{x^{\frac{9}{2}}}{\frac{9}{2}} - 2\dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} + 3\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \dfrac{2}{9}x^{\frac{9}{2}} - 2\times\dfrac{2}{5}x^{\frac{5}{2}} + \cancel{3}\times\dfrac{2}{\cancel{3}}x^{\frac{3}{2}} + c$

$= \dfrac{2}{9}x^{\frac{9}{2}} - \dfrac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + c$

5. $\dfrac{8x+13}{\sqrt{4x+7}}$

Sol : $\int \dfrac{8x+13}{\sqrt{4x+7}}dx$

$= \int \dfrac{8x+14-1}{\sqrt{4x+7}}dx = \int \dfrac{2(4x+7)-1}{\sqrt{4x+7}}dx$

$= 2\int \dfrac{(4x+7)}{\sqrt{4x+7}}dx - \int \dfrac{1}{\sqrt{4x+7}}dx$

$= 2\int \sqrt{4x+7}\,dx - \int \dfrac{1}{\sqrt{4x+7}}dx$

$= 2\int (4x+7)^{\frac{1}{2}}dx - \int (4x+7)^{-\frac{1}{2}}dx$

$= 2\dfrac{(4x+7)^{\frac{1}{2}+1}}{4\left(\frac{1}{2}+1\right)} - \dfrac{(4x+7)^{-\frac{1}{2}+1}}{4\left(\frac{-1}{2}+1\right)} + c$

$= 2\dfrac{(4x+7)^{\frac{3}{2}}}{4\left(\frac{3}{2}\right)} - \dfrac{(4x+7)^{\frac{1}{2}}}{4\left(\frac{1}{2}\right)} + c$

$= \cancel{2}\dfrac{(4x+7)^{\frac{3}{2}}}{\cancel{6}\,3} - \dfrac{(4x+7)^{\frac{1}{2}}}{2} + c$

$= \dfrac{(4x+7)^{\frac{3}{2}}}{3} - \dfrac{(4x+7)^{\frac{1}{2}}}{2} + c$

6. $\dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}$

Sol : $\int \dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}dx$

Multiplying and dividing the conjugate of the denominator we get

$= \int \dfrac{\sqrt{x+1}-\sqrt{x-1}\,dx}{\left(\sqrt{x+1}+\sqrt{x-1}\right)\left(\sqrt{x+1}-\sqrt{x-1}\right)}$

$= \int \dfrac{\sqrt{x+1}-\sqrt{x-1}}{(x+1)-(x-1)}\,dx$

[∵ $(a+b)(a-b) = a^2-b^2$]

$= \int \dfrac{\sqrt{x+1}-\sqrt{x-1}}{\cancel{x}+1-\cancel{x}+1}\,dx = \int \dfrac{\sqrt{x+1}-\sqrt{x-1}}{2}\,dx$

$= \dfrac{1}{2}\int ((x+1)^{½}-(x-1)^{½})\,dx$

$= \dfrac{1}{2}\left[\dfrac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{(x-1)^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$

$= \dfrac{1}{\cancel{2}\times\frac{3}{\cancel{2}}}\left[(x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}}\right] + c$

$= \dfrac{1}{3}\left[(x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}}\right] + c$

7. If $f'(x) = x + b$, $f(1) = 5$ and $f(2) = 13$, then find $f(x)$

Sol : Given $f'(x) = x + b$, $f(1) = 5$ and $f(2) = 13$

$f'(x) = x + b \Rightarrow \int f'(x)\,dx = \int (x+b)\,dx$

[∴ Integration is the reverse process of differentiation]

$\Rightarrow f(0) = \dfrac{x^2}{2} + bx + c$ ...(1)

Given $f(1) = 5$

$\Rightarrow 5 = \dfrac{1^2}{2} + b(1) + c$

$\Rightarrow 5 = \dfrac{1}{2} + b(1) + c \Rightarrow 5 - \dfrac{1}{2} = b + c$

$\Rightarrow \dfrac{10-1}{2} = b + c \Rightarrow b + c = \dfrac{9}{2}$

$\Rightarrow 2b + 2c = 9$ ...(2)

Also $f(2) = 13 \Rightarrow \quad 13 = \frac{2^2}{2} + b(2) + c$

$\Rightarrow \quad 13 = 2 + 2b + c$

$\Rightarrow \quad 13 - 2 = 2b + c$

$\Rightarrow \quad 2b + c = 11 \quad \ldots(3)$

$(2) - (3) \rightarrow 2b + 2c = 9$

$\quad -2b + -c = -11$

$\boxed{c = -2}$

Substituting $c = -2$ in (3) we get

$2b - 2 = 11 \Rightarrow 2b = 11 + 2 \Rightarrow 2b = 13$

$\Rightarrow \quad \boxed{b = \frac{13}{2}}$

Substituting $b = \frac{13}{2}$, $c = -2$ in (1) we get,

$$f(x) = \frac{x^2}{2} + \frac{13}{2}x - 2$$

8. If $f'(x) = 8x^3 - 2x$ and $f(2) = 8$, then find $f(x)$.

Sol : Given $f'(x) = 8x^3 - 2x$, $f(2) = 8$

$f'(x) = 8x^3 - 2x$

$\Rightarrow \int f'(x)\,dx = \int (8x^3 - 2x)\,dx$

$\Rightarrow \quad f(x) = \frac{\overset{2}{\cancel{8}}x^4}{\cancel{4}} - \frac{\cancel{2}x^2}{\cancel{2}} + c$

$\Rightarrow \quad f(x) = 2x^4 - x^2 + c \quad \ldots(1)$

Given $f(2) = 8$

$\Rightarrow \quad 8 = 2(2^4) - 2^2 + c$

$\Rightarrow \quad 8 = 32 - 4 + c \Rightarrow 8 = 32 - 4 + c$

$\Rightarrow \quad 8 - 28 = c$

$\Rightarrow \quad c = -20$

Substituting $c = -20$ in (1) we get.

$$f(x) = 2x^4 - x^2 - 20$$

EXERCISE 2.2

Integrate the following with respect to x.

1. $\left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right)^2$

Sol : $\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right)^2 dx$

$= \int \left[\left(\sqrt{2x}\right)^2 - 2\left(\cancel{\sqrt{2x}}\right)\left(\frac{1}{\cancel{\sqrt{2x}}}\right) + \left(\frac{1}{\sqrt{2x}}\right)^2\right] dx$

$[\because (a-b)^2 = a^2 - 2ab + b^2]$

$= \int \left(2x - 2 + \frac{1}{2x}\right) dx$

$= \cancel{2}\frac{x^2}{\cancel{2}} - 2x + \frac{1}{2}\log|x| + c$

$= x^2 - 2x + \frac{1}{2}\log|x| + c$

2. $\frac{x^4 - x^2 + 2}{x - 1}$

Sol : $\int \frac{x^4 - x^2 + 2}{x - 1}\,dx$

$= \int \left(x^3 + x^2 + \frac{2}{x - 1}\right) dx$

$= \frac{x^4}{4} + \frac{x^3}{3} + 2\log|x-1| + c$

3. $\frac{x^3}{x + 2}$

Sol : $\int \frac{x^3}{x + 2}\,dx = \int \left(x^2 - 2x + 4 - \frac{8}{x + 2}\right) dx$

$= \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + 4x - 8\log|x + 2| + c$

$= \frac{x^3}{3} - x^2 + 4x - 8\log|x + 2| + c$

$[\because \int \frac{1}{x}dx = \log|x| + c]$

```
        x^2 - 2x + 4
x+2 | x^3
      (-)x^3 +(-)2x^2
      ------------
         -2x^2
      (+)-2x^2 (+)-4x
      ------------
                4x
          (-)4x +(-)8
          --------
               -8
```

4. $\frac{x^3 + 3x^2 - 7x + 11}{x + 5}$

Sol : $\int \frac{x^3 + 3x^2 - 7x + 11}{x + 5}\,dx$

$= \int \left(x^2 - 2x + 3 - \frac{4}{x + 5}\right) dx$

$= \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + 3x - 4\log|x + 5| + c$

$= \frac{x^3}{3} - x^2 + 3x - 4\log|x + 5| + c$

$x^2 - 2x + 3$

$x+5 \,)\, x^3 + 3x^2 - 7x + 11$

$\quad {}^{(-)}x^3 + {}^{(-)}5x^2$

$\quad -2x^2 - 7x$

$\quad 2x^2 + 10x$

$\quad (-) \quad (-)$

$\quad 3x + 11$

$\quad 3x + 15$

$\quad (-) \quad (-)$

$\quad -4$

$^{(-)}x^3 + ^{(-)}5x^2$

5. $\dfrac{\mathbf{3x+2}}{\mathbf{(x-2)(x-3)}}$ **[Qy - 2019]**

Sol : $\displaystyle\int \frac{(3x+2)\,dx}{(x-2)(x-3)} = \int\left(\frac{-8}{x-2} + \frac{11}{x-3}\right)dx$

$= -8 \log|x-2| + 11 \log|x-3| + c$

$= 11 \log|x-3| - 8 \log|x-2| + c$

$\dfrac{3x+2}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$

$\Rightarrow 3x + 2 = A(x-3) + B(x-2)$

Put $x = 3$

$9 + 2 = B(1) \Rightarrow B = 11$

Put $x = 2$

$8 = A(-1) \Rightarrow A = -8$

$\dfrac{3x+2}{(x-2)(x-3)} = \dfrac{-8}{x-2} + \dfrac{11}{x-3}$

6. $\dfrac{\mathbf{4x^2+2x+6}}{\mathbf{(x+1)^2(x-3)}}$ **[HY-2019]**

Sol : $\displaystyle\int \frac{4x^2+2x+6}{(x+1)^2(x-3)}\,dx$

$= \displaystyle\int\left(\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}\right)dx$

$= \displaystyle\int\left(\frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{3}{x-3}\right)dx$

$= \log|x+1| - 2\int (x+1)^{-2}\,dx + 3\log|x-3| + c$

$= \log|x+1| - 2\,\dfrac{(x+1)^{-2+1}}{-2+1} + 3\log|x-3| + c$

$= \log|x+1| + 2(x+1)^{-1} + 3\log|x-3| + c$

$= \log|x+1| + \dfrac{2}{x+1} + 3\log|x-3| + c$

$\dfrac{4x^2+2x+6}{(x+1)^2(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-3}$

$4x^2 + 2x + 6 = A(x+1)(x-3) + B(x-3) + C(x+1)^2$

$\Rightarrow$ Putting $x = -1$, $4 - 2 + 6 = B(-4)$

$\Rightarrow 8 = B(-4) \Rightarrow B = -2$

Putting $x = 3$

$36 + 6 + 6 = C(16)$

$\Rightarrow 48 = 16C \Rightarrow C = 3$

Putting $x = 0$,

$6 = -3A - 3B + C$

$\Rightarrow 6 = -3A + 6 + 3$

$\Rightarrow 3A = 3 \Rightarrow A = 1$

$x^3 + x^2$

$x-1 \,)\, x^4 - x^2 + 2$

$\quad (-) \quad (+)$

$\quad x^4 - x^3$

$\quad x^3 - x^2$

$\quad (-) \quad (+)$

$\quad x^3 - x^2$

$\quad 2$

7. $\dfrac{\mathbf{3x^2-2x+5}}{\mathbf{(x-1)(x^2+5)}}$ **[Sep. - 2020; Aug. - 2021]**

Sol : $\displaystyle\int \frac{3x^2-2x+5}{(x-1)(x^2+5)}\,dx = \int\left(\frac{A}{x-1} + \frac{Bx+c}{x^2+5}\right)dx$

$= \displaystyle\int\left(\frac{1}{x-1} + \frac{2x+0}{x^2+5}\right)dx = \int\frac{1}{x-1}\,dx + \int\frac{2x}{x^2+5}\,dx$

$= \log|x-1| + \log|x^2+5| + c$

$\left[\because \displaystyle\int\frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$

$= \log|(x^2+5)(x-1)| + c$

$[\because \log m + \log n = \log mn]$

$= \log|x^3 - x^2 + 5x - 5| + c$

$\dfrac{3x^2-2x+5}{(x-1)(x^2+5)} = \dfrac{A}{(x-1)} + \dfrac{Bx+c}{(x^2+5)}$

$\Rightarrow 3x^2 - 2x + 5 = A(x^2+5) + (Bx + C)(x-1)$

Putting $x = 1$,

$3 - 2 + 5 = A(1+5)$

$\Rightarrow 6 = A(6) \Rightarrow$ $\boxed{A = 1}$

Putting $x = 0$,

$5 = 5A - C$

$\Rightarrow 5 = 5 - C$ $[\because A = 1]$

$\Rightarrow C = 5 - 5 \Rightarrow$ $\boxed{C = 0}$

Putting $x = -1$,

$3 + 2 + 5 = A(6) + (C - B)(-2)$

$\Rightarrow 10 = 6A + 2B - 2C$

$\Rightarrow 10 = 6 + 2B + 0$

$\Rightarrow 10 - 6 = 2B \Rightarrow 4 = 2B$

$\Rightarrow$ $\boxed{B = 2}$

8. If $f'(x) = \frac{1}{x}$ and $f(1) = \frac{\pi}{4}$, then find $f(x)$.

Sol : Given $f'(x) = \frac{1}{x}$ [PTA - 2]

$\Rightarrow \int f'(x)\,dx = \int \frac{1}{x}\,dx$

$\Rightarrow f(x) = \log|x| + c$ ...(1)

Also, $f(1) = \frac{\pi}{4}$, we get

$\Rightarrow \frac{\pi}{4} = \log|1| + c$

$\Rightarrow \frac{\pi}{4} = c$ $[\because \log 1 = 0]$

Substituting $c = \frac{\pi}{4}$ in (1) we get,

$f(x) = \log|x| + \frac{\pi}{4}$

EXERCISE 2.3

Integrate the following with respect to x

1. $e^{x\log a} + e^{a\log a} - e^{n\log x}$

Sol : $\int(e^{x\log a} + e^{a\log a} - e^{n\log x})\,dx$

$= \int\left(e^{\log a^x} + e^{\log a^a} - e^{\log x^n}\right)dx$ $[\because m\log n = \log n^m]$

$= \int(a^x + a^a - x^n)\,dx$ $[\because e^{\log x} = x]$

$= \int a^x\,dx + \int a^a\,dx - \int x^n dx$

$= \left[\frac{a^x}{\log a}\right] + a^a(x) - \frac{x^{n+1}}{n+1} + c$ $\left[\because \int a^x = \frac{a^x}{\log a}\right]$

2. $\frac{a^x - e^{x\log b}}{e^{x\log a}\, b^x}$

Sol : $\int \frac{a^x - e^{x\log b}}{e^{x\log a}\, b^x}\,dx$

$= \int \frac{a^x - e^{\log b^x}}{e^{\log a^x}\, b^x}\,dx$ $[\because m\log n = \log n^m]$

$= \int \frac{a^x - b^x}{a^x . b^x}\,dx$ $[\because e^{\log x} = x]$

$= \int \frac{a^x}{a^x b^x}\,dx - \int \frac{b^x}{a^x b^x}\,dx$

$= \int \frac{1}{b^x}\,dx - \int \frac{1}{a^x}\,dx$

$= \int b^{-x}\,dx - \int a^{-x}\,dx$ $\left[\because \int a^{-x}\,dx = \frac{a^{-x}}{-\log a} + c\right]$

$= \frac{b^{-x}}{-\log b} - \frac{a^{-x}}{-\log a} + c$

$= -\frac{b^{-x}}{\log b} + \frac{a^{-x}}{\log a} + c$

$= \frac{-1}{b^x \log b} + \frac{1}{\log a . a^x} + c$

$= \frac{1}{a^x \log a} - \frac{1}{b^x \log b} + c$

3. $(e^x + 1)^2 e^x$

Sol : $\int (e^x+1)^2 e^x\,dx = \int [(e^x)^2 + 2(e^x)(1) + 1^2]\, e^x\,dx$ $[\because (a+b)^2 = a^2 + 2ab + b^2]$

$= \int (e^{2x} + 2e^x + 1)\, e^x\,dx$

$= \int (e^{3x} + 2e^{2x} + e^x)\,dx$ $[a^m . a^n = a^{m+n}]$

$= \int e^{3x}\,dx + 2\int e^{3x}\,dx + \int e^x\,dx$

So $\int (e^x + 1)^2 e^x\,dx = \int e^{3x}\,dx + \int 2e^{2x}\,dx + \int e^x\,dx$

$= \frac{e^{3x}}{3} + 2\frac{e^{2x}}{2} + e^x + c$

$= e^x + e^{2x} + \frac{e^{3x}}{3} + c$

4. $\frac{e^{3x} - e^{-3x}}{e^x}$ [Aug. - 2021]

Sol : $\int \frac{e^{3x} - e^{-3x}}{e^x}\,dx = \int \frac{e^{3x}}{e^x}\,dx - \int \frac{e^{-3x}}{e^x}\,dx$

$= \int e^{3x-x}\,dx - \int e^{-3x-x}\,dx$ $\left[\because \frac{a^m}{a^m} = a^{m-n}\right]$

$= \int e^{2x}\,dx - \int e^{-4x}\,dx$

$= \frac{e^{2x}}{2} - \frac{e^{-4x}}{-4} + c = \frac{e^{2x}}{2} + \frac{e^{-4x}}{4} + c$

5. $\frac{e^{3x} + e^{5x}}{e^x + e^{-x}}$

Sol : $\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}}\,dx = \int \frac{e^{4x-x} + e^{4x+x}}{e^x + e^{-x}}$

$= \int \frac{e^{4x}(e^{-x} + e^x)}{e^x + e - x}\,dx$

So $\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}} = \int e^{4x}\,dx = \frac{e^{4x}}{4} + c$

6. $\left(1-\frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)}$

Sol : Let $I = \int\left(1-\frac{1}{x^2}\right) . e^{\left(x+\frac{1}{x}\right)} dx$

put $x + \frac{1}{x} = t$

on differentiating we get, $\left(1-\frac{1}{x^2}\right) dx = dt$

$\therefore \quad I = \int e^t dt = e^t + c$

$= e^{x+\frac{1}{x}} + c \qquad [\because t = x + \frac{1}{x}]$

7. $\frac{1}{x(\log x)^2}$

Sol : Let $I = \int\frac{1}{x(\log x)^2} dx$

put $\log x = t$ on differentiating we get,

$\frac{1}{x} dx = dt$

$\therefore \quad I = \int\frac{1}{t^2} dt \qquad [\because t = \log x]$

$I = \int t^{-2} dt = \frac{t^{-2+1}}{-2+1} + c$

$= \frac{t^{-1}}{-1} + c \Rightarrow \frac{-1}{t} + c$

$= \frac{-1}{\log|x|} + c \qquad [\because t = \log |x|]$

8. If $f'(x) = e^x$ and $f(0) = 2$, then find $f(x)$

Sol : Given $f'(x) = e^x$

$\Rightarrow \int f'(x) dx = \int e^x dx$

[Taking integration both sides]

$\Rightarrow f(x) = e^x + c \qquad ...(1)$

Also, $f(0) = 2$

$\Rightarrow 2 = e^0 + c$

$\Rightarrow 2 = 1 + c$

$\Rightarrow 2 - 1 = c$

$\Rightarrow c = 1$

Substituting $c = 1$ in (1) we get,

$f(x) = e^x + 1$

EXERCISE 2.4

Integrate the following with respect to x.

1. $2\cos x - 3\sin x + 4\sec^2 x - 5\operatorname{cosec}^2 x$

Sol : $\int (2\cos x - 3\sin x + 4\sec^2 x - 5\operatorname{cosec}^2 x)\, dx$

$= 2\int\cos x\, dx - 3\int\sin x\, dx + 4\int\sec^2 x\, dx - 5\int\operatorname{cosec}^2 x\, dx$

$= 2(\sin x) - 3(-\cos x) + 4\tan x - 5(-\cot x) + c$

$= 2\sin x + 3\cos x + 4\tan x + 5\cot x + c$

2. $\sin^3 x$

Sol : $\int \sin^3 x\, dx$

We know that $\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\therefore \int \sin^3 x\, dx = \frac{1}{4}\int(3\sin x - \sin 3x)\, dx$

$= \frac{3}{4}\int\sin x\, dx - \frac{1}{4}\int\sin 3x\, dx$

$= \frac{3}{4}(-\cos x) - \frac{1}{4}\left(-\frac{\cos 3x}{3}\right) + c$

$[\because \int\sin ax\, dx = \frac{-1}{a}\cos ax + c]$

$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + c$

3. $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$ **[March - 2020]**

Sol : $\int\frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

$= \int\frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x} dx = \sec^2 x$

$= \int\frac{\cos 2x + 2\sin^2 x}{\cos^2 x} \qquad [\because \cos 2x = \cos^2 x - \sin^2 x]$

$= \int\frac{\cos^2 x + \sin^2 x}{\cos^2 x} dx$

$= \int\frac{1}{\cos^2 x} dx = \sec^2 x \qquad [\because \sin^2 x + \cos^2 x = 1]$

So $\int\frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \int\sec^2 x\, dx = \tan x + c$

4. $\frac{1}{\sin^2 x\cos^2 x}$ [Hint: $\sin^2 x + \cos^2 x = 1$]

Sol : $\int\frac{1}{\sin^2 x\cos^2 x} dx$

$= \int\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x\cos^2 x} dx \qquad [\because 1 = \sin^2 x + \cos^2 x]$

$= \int\frac{\sin^2 x}{\sin^2 x\cos^2 x} dx + \int\frac{\cos^2 x}{\sin^2 x\cos^2 x} dx$

Chapter 3

INTEGRAL CALCULUS-II

SNAPSHOT

Geometrical interpretation of definite integral is the area under a curve between the given limits.
Integration helps us to find out the total cost function and total revenue function from the marginal cost.
Consumer's surplus & producer's surplus theory was developed by the economist Marshal.

FORMULAE TO REMEMBER

1. Area of the region bounded by $y = f(x)$ between the limits $x = a$, $x = b$ and less below x axis is

$$A = \int_a^b -y\,dx = -\int_a^b f(x)\,dx$$

2. Area of the region bounded by $x = f(y)$, between the limits $y = c$, $y = d$ with Y - axis and the area lies to the right of Y-axis is

$$A = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

3. Area bounded by $x = f(y)$ between the limits $y = c$, $y = d$ with Y-axis and the area lies to the left of Y-axis is

$$A = \int_c^d -x\,dy = -\int_c^d f(y)\,dy$$

4. Area between two curves from $x = a$ to $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

5. If c is the cost function, marginal cost function $MC = \frac{dC}{dx}$

6. $C = \int (MC)\,dx + k$

7. Average cost function $AC = \frac{C}{x}, x \neq 0$

8. If R is the total revenue function, marginal revenue function $MR = \frac{dR}{dx}$

9. $R = \int (MR)\,dx + k$

10. Demand function $P = \frac{R}{x}, x \neq 0$

11. If 'P' denotes the profit function, then $P = \int (MR - MC)\,dx + k$

12. Total inventory carrying cost $= C_1 \int_0^T I(x)\,dx$,where
C1 → holding cost
T → time period
I(x) → Inventory on hand.

13. Amount of annuity after N payments $A = \int_0^N pe^{rt}\,dt$

14. Elasticity of demand $\eta_d = \frac{-p}{x}\frac{dx}{dp}$

15. $\frac{Ey}{Ex} = \frac{x}{y}\frac{dy}{dx}$

16. Consumer's surplus CS $= \int_0^{x_0} f(x)\,dx - x_0 p_0$
where $f(x)$ is the demand function.

17. Producer's surplus PS $= x_0 p_0 - \int_0^{x_0} g(x)\,dx$
where $g(x)$ is the supply function.

TEXTUAL QUESTIONS

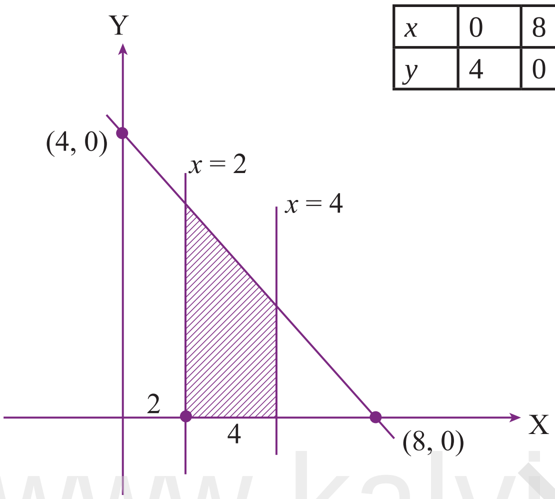
EXERCISE 3.1

1. Using Integration, find the area of the region bounded the line $2y + x = 8$, the x axis and the lines $x = 2, x = 4$. [PTA - 6]

Sol:

$2y + x = 8$

x	0	8
y	4	0



$$\text{Given} \quad 2y + x = 8$$
$$2y = 8 - x$$
$$y = \frac{1}{2}(8 - x)$$

Given limits are $x = 2$ and $x = 4$

Area of the shaded region between the given limits

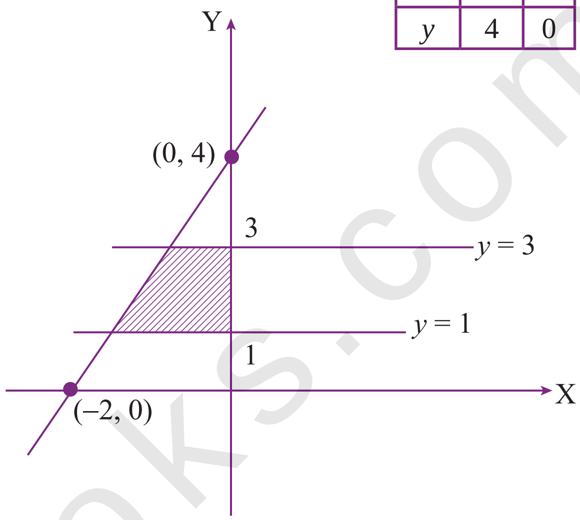
$$A = \int_a^b y\,dx = \int_2^4 \frac{1}{2}(8-x)dx$$
$$\frac{1}{2}\int_2^4 (8-x)dx = \frac{1}{2}\left[8x - \frac{x^2}{x}\right]_2^4$$
$$= \frac{1}{2}\left[\left(8(4) - \frac{4^2}{2}\right) - \left(8(2) - \frac{2^2}{2}\right)\right]$$
$$= \frac{1}{2}[(32-8)-(16-21)]$$
$$= \frac{1}{2}[24-14]$$
$$= \frac{y-4}{2}(10)$$
$$A = 5 \text{ sq. units.}$$

2. Find the area bounded by the lines $y - 2x - 4 = 0$, $y = 1$, $y = 3$ and the y-axis [HY-2019]

Sol:

$y - 2x - 4 = 0$

x	0	–2
y	4	0



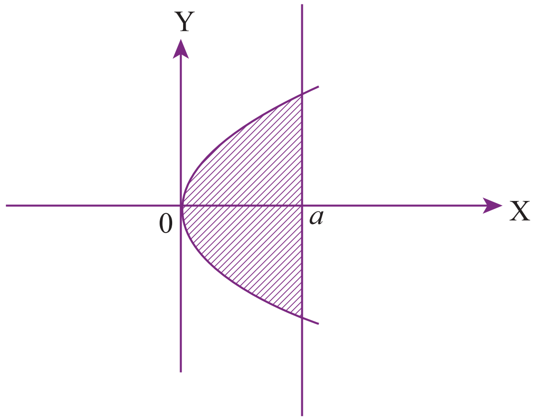
$$\text{Given } y - 2x - 4 = 0$$
$$\Rightarrow \quad y - 4 = 2x$$
$$\Rightarrow \quad x = \frac{y-4}{2}$$

Since the area lies to the left of Y-axis, with the limits $y = 1$ & $y = 3$.

$$\text{Area} = \int_1^3 -x\,dy$$
$$= \int_1^3 -\left(\frac{1}{2}\right)(y-4)dy$$
$$= \frac{1}{2}\int_1^3 (4-y)dy = \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_1^3$$
$$= \frac{1}{2}\left[\left(4(3) - \frac{3^2}{2}\right) - \left(4(1) - \frac{1^2}{2}\right)\right]$$
$$= \frac{1}{2}\left[\left(12 - \frac{9}{2}\right) - \left(4 - \frac{1}{2}\right)\right]$$
$$= \frac{1}{2}\left[\left(\frac{24-9}{2}\right) - \left(\frac{8-1}{2}\right)\right]$$
$$= \frac{1}{2}\left[\frac{15}{2} - \frac{7}{2}\right] = \frac{1}{2}\left[\frac{8}{2}\right]$$
$$= \frac{y-4}{2}$$
$$A = 2 \text{ sq. units.}$$

3. Calculate the area bounded by the parabola $y^2 = 4\,ax$ and its latus rectum. [Sep. - 2020]

Sol:



$$y^2 = 4\,ax \text{ is the right open}$$
$$\Rightarrow \quad y = \sqrt{4ax} \text{ parabola.}$$

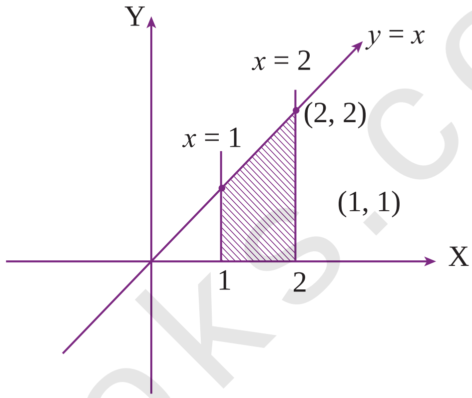
The limits are from $x = 0$ to $x = a$

$$\therefore \text{Area} = 2\int_0^a y\,dx = 2\int_0^a \sqrt{4ax}\,dx$$
$$= 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx$$
$$= 4\sqrt{a}\int_0^a \sqrt{x}\,dx$$
$$= 4\sqrt{a}\int_0^a x^{\frac{1}{2}}dx$$
$$= 4\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$
$$= 4\sqrt{a}\times\frac{2}{3}\left(x^{\frac{3}{2}}\right)_0^a$$
$$= 8\frac{\sqrt{a}}{3}\left(a^{\frac{3}{2}} - 0\right)$$
$$= \frac{8\sqrt{a}}{3}\left(a\sqrt{a}\right)$$
$$= \frac{8}{3}a^2 \text{ sq. units}$$

4. Find the area bounded by the line $y = x$, the x-axis and the ordinates $x = 1, x = 2$. [PTA-4;QY-2019]

Sol: $y = x$

x	1	2
y	1	2



Given line $y = x$

The limits are $x = 1, x = 2$

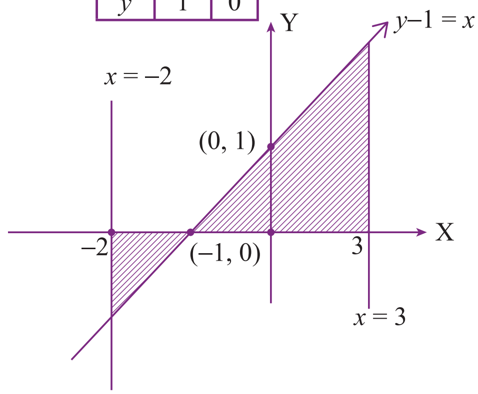
Since the shaded area lies to the right of Y-axis,

$$\text{Area} = \int_1^2 y\,dx = \int_1^2 x\,dx = \left(\frac{x^2}{2}\right)_1^2$$
$$= \frac{2^2}{2} - \frac{1^2}{2} = 2 - \frac{1}{2}$$
$$= \frac{4-1}{2} = \frac{3}{2} \text{ sq. units.}$$

5. Using integration, find the area of the region bounded by the line $y - 1 = x$, the x-axis and the ordinates $x = -2, x = 3$.

Sol: $y - 1 = x$

x	0	–1
y	1	0



Given line is $y - 1 = x \Rightarrow y = x + 1$

Given limits are from $x = -2$ to 3.

In the diagram, the area from $x = -2$ to $x = -1$ lies below the X-axis and the area from $x = -1$ to $x = 3$ lies above the X-axis.

∴ Required Area

$$= \int_{-2}^{-1} -y\,dx + \int_{-1}^{3} y\,dx = -\int_{-2}^{-1} y\,dx + \int_{-1}^{3} y\,dx$$

$$= \int_{-1}^{-2} y\,dx + \int_{-1}^{3} y\,dx \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \int_{-1}^{-2} (x+1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

$$= \left(\frac{x^2}{2}+x\right)_{-1}^{-2} + \left(\frac{x^2}{2}+x\right)_{-1}^{3}$$

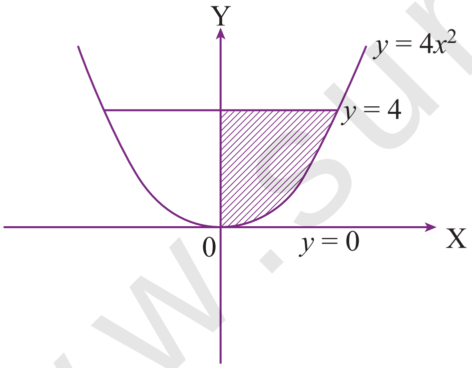
$$= \left(\frac{4}{2}-2\right) - \left(\frac{1}{2}-1\right) + \left(\frac{9}{2}+3\right) - \left(\frac{1}{2}-1\right)$$

$$= (2-2) - \left(-\frac{1}{2}\right) + \left(\frac{15}{2}\right) - \left(-\frac{1}{2}\right)$$

$$= 0 + \frac{1}{2} + \frac{15}{2} + \frac{1}{2} = \frac{17}{2} \text{ sq. units.}$$

6. Find the area of the region lying in the first quadrant bounded by the region $y = 4x^2$, $x = 0$, $y = 0$ and $y = 4$.

Sol:



Given curve $y = 4x^2$ is an open upward parabola

$$\Rightarrow \quad \frac{y}{4} = x^2$$

The limits are from $y = 0$ to $y = 4$

Since the shaded region lies to the right of Y-axis,

$$\text{Required area} = \int_0^4 x\,dy = \int_0^4 \sqrt{\frac{y}{4}}\,dy$$

$$= \frac{1}{2}\int_0^4 \sqrt{y}\,dy = \frac{1}{2}\int_0^4 y^{\frac{1}{2}}\,dy$$

$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \frac{1}{2} \times \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^4$$

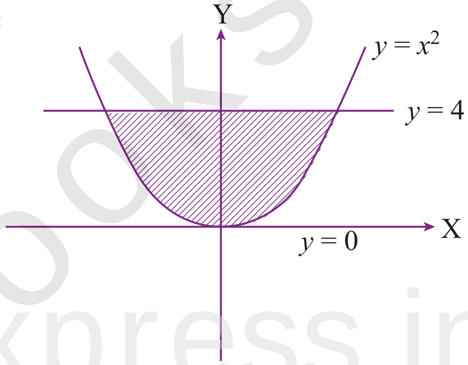
$$= \frac{1}{3}\left[4^{\frac{3}{2}} - 0\right] = \frac{1}{3}\left[4^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\,4^1\sqrt{4} = \frac{1}{3}\cdot 4(2)$$

$$A = \frac{8}{3} \text{ sq. units.}$$

7. Find the area bounded by the curve $y = x^2$ and the line $y = 4$. **[Aug. - 2021]**

Sol:



Since $y = x^2$ is symmetric about Y-axis, the required

$$\text{Area} = 2\int_0^4 x\,dy$$

When $y = x^2 \Rightarrow x = \sqrt{y}$

$$\therefore \quad \text{Area} = 2\int_0^4 \sqrt{y}\,dy = 2\int_0^4 y^{\frac{1}{2}}\,dy$$

$$= 2\left[\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^4$$

$$= 2 \times \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^4$$

$$= \frac{4}{3}\left[4^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{4}{3}\left[4^{\frac{3}{2}}\right]$$

$$= \frac{4}{3} \times \left(2^2\right)^{\frac{3}{2}} = \frac{4}{3} \times 2^3$$

$$A = \frac{4}{3} \times 8 = \frac{32}{3} \text{ sq. units.}$$

Chapter 4

DIFFERENTIAL EQUATIONS

SNAPSHOT

- An ordinary differential equation is an equation that involves some ordinary derivatives $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\right)$ of a function $y = f(x)$. Here we have one independent variable.
- Sometimes a family of curves can be represented by a single equation with one or move arbitrary constants. By assigning different values for constants, we get a family of curves. The arbitrary constants are called the parameters of the family.
- Solution of the differential equation must contain the same number of arbitrary constants as the order of the equation. Such a solution is called General (complete) solution of the differential equation.
- The highest order derivative present in the differential equation is the order of the differential equation.
- Degree is the highest power of the highest order derivative in the differential equation.
- Variable separable method

  If in an equation, it is possible to collect all the terms of x and dx on one side and all the terms of y and dy on the other side, then the variable are said to be separable $f(x)dx = g(y)dy$.
- Homogeneous differential equations:

  $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is homogeneous differential equation if $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree in x and y.
- Linear differential equations of first order:

  General form of linear equation of first order is $\frac{dy}{dx} + Py = Q$.
- Second order first degree differential equations

  $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ is the general form of second order first degree differential equations.

POINTS TO REMEBER

- Variable separable method

  Separate the variable x and y and then integrate.
- Homogeneous differential equations

  Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

  Then separate the variables v and x, integrate and finally put $v = \frac{y}{x}$
- Linear differential equations of first order
- Solution of $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x is

  $y\, e^{\int pdx} = \int Q\, e^{\int pdx}\, dx + c.$

  Also $e^{\int pdx}$ is known as Integrating Factor (I.F).
- Second order first degree differential equations
- If $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ then y = CF + PI

  [CF → Complementary function; PI → Particular integral]

Solution of CF

Nature of roots	Complementary function (CF)
1) Real and different ($m_1 \neq m_2$)	$Ae^{m_1x} + Be^{m_2x}$
2) Real and equal ($m_1 = m_2 = m$)	$(Ax + B)e^{mx}$
3) Complex roots ($\alpha \pm i\beta$)	$e^{\alpha x}[A \cos \beta x + B \sin \beta x]$
	Where A and B are constants

Solution of Particular Intergral (P.I)

If $f(x) = e^{ax} \Rightarrow \phi(D)y = e^{ax}$

PI = $\frac{1}{\phi(D)}e^{ax}$ $\phi(D) \neq 0$

If $\phi(D) \neq 0$, when $D = a$, then PI = $x \cdot \frac{1}{\phi^1(D)}e^{ax}$

If $\phi^1(D) \neq 0$, when $D = a$, then PI = $x^2 \cdot \frac{e^{ax}}{\phi^1(D)}$

TEXTUAL QUESTIONS

EXERCISE 4.1

1. Find the order and degree of the following differential equations.

(i) $\frac{dy}{dx}+2y=x^3$

(ii) $\left(\frac{d^3y}{dx^3}\right)+3\left(\frac{dy}{dx}\right)^3+2\left(\frac{dy}{dx}\right)=0$

(iii) $\frac{d^2y}{dx^2}=\sqrt{y-\frac{dy}{dx}}$

(iv) $\frac{d^3y}{dx^3}=0$

(v) $\frac{d^2y}{dx^2}+y+\left(\frac{dy}{dx}-\frac{d^3y}{dx^3}\right)^{\frac{3}{2}}=0$

(vi) $(2-y'')^2 = y''^2+2y'$

(vii) $\left(\frac{dy}{dx}\right)^3+y=x-\frac{dx}{dy}$

Sol. (i) The highest derivative is first order and its power is one

$\therefore$ order: $\rightarrow 1$, degree: $\rightarrow 1$

(ii) The highest derivate is third order and its power is one.

$\therefore$ order: $\rightarrow 3$, degree: $\rightarrow 1$

(iii) Squaring both sides, we get $\left(\frac{d^2y}{dx^2}\right)^2=y-\frac{dy}{dx}$

The highest derivative is second order and its power is 2.

$\therefore$ Order is 2 and degree is 2.

(iv) The highest derivative is of third order and its power is 1. Order is 3 and degree is 1.

(v) $\frac{d^2y}{dx^2}+y=-\left(\frac{dy}{dx}-\frac{d^3y}{dx^3}\right)^{\frac{3}{2}}$

Squaring both sides we get,

$$\left[\left(\frac{d^2y}{dx^2}\right)^2+y\right]^2=\left[-\left(\frac{dy}{dx}\right)-\frac{d^3y}{dx^3}\right]^{\frac{3}{2}\times 2}$$

$$\left(\frac{d^2y}{dx^2}\right)^4+y^2+2xy\left(\frac{d^2y}{dx^2}\right)=\left(\frac{dy}{dx}-\frac{d^3y}{dx^3}\right)^3$$

The highest derivate is of third order and its power is 3.

$\therefore$ Order is 3 and degree is 3.

(vi) $(2-y'')^2 = y''^2+2y'$

$\Rightarrow 4+\cancel{y''^2}-4y'' = \cancel{y''^2}+2y'$

$\Rightarrow 4-4y'' = 2y'$

The highest derivative is of second order and its power is 1.

$\therefore$ Order is 2 and degree is 1.

(vii) $\left(\frac{dy}{dx}\right)^3+y = x-\frac{1}{\left(\frac{dy}{dx}\right)}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3+y = \frac{x\left(\frac{dy}{dx}\right)-1}{\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^4+y\left(\frac{dy}{dx}\right) = x\left(\frac{dy}{dx}\right)-1$$

The highest derivative is of first order and its power is 3.

$\therefore$ Order is 1 and degree is 4.

2. Find the differential equation of the following:

(i) $y = cx+c-c^3$ **(ii)** $y = c(x-c)^2$

(iii) $xy = c^2$ **(iv)** $x^2+y^2=a^2$

Sol. (i) Given equation is $y = cx+c-c^3$ ...(1)

Differentiating with respect to 'x' we get,

$\frac{dy}{dx}=c(1)+0-0 \Rightarrow \frac{dy}{dx}=c$ ...(2)

Substituting (2) in (1) we get,

$y = x\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)-\left(\frac{dy}{dx}\right)^3$ which is the required differential equation.

(ii) Given equation is $y = c(x-c)^2$ ...(1)

Differentiating with respect to 'x' we get

$\frac{dy}{dx}=2c(x-c)$ ...(2)

(1) ÷ (2) gives

$$\frac{y}{\frac{dy}{dx}}=\frac{c(x-c)^2}{2c(x-c)} \Rightarrow \frac{y}{\frac{dy}{dx}}=\frac{x-c}{2}$$

$$x-c=\frac{2y}{\frac{dy}{dx}} \Rightarrow c = x-\frac{2y}{\frac{dy}{dx}}$$

Substituting the value of c in (1) we get

$$y = \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left(x - x + \frac{2y}{\frac{dy}{dx}}\right)^2$$

$$= \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left(\frac{4y^2}{\left(\frac{dy}{dx}\right)^2}\right)$$

$$y = \left(\frac{x\frac{dy}{dx} - 2y}{\frac{dy}{dx}}\right)\left(\frac{4y^2}{\left(\frac{dy}{dx}\right)^2}\right)$$

$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^3 = 4y^2\left(x\frac{dy}{dx} - 2y\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 = 4xy\left(\frac{dy}{dx}\right) - 8y^2$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 - 4xy\left(\frac{dy}{dx}\right) + 8y^2 = 0$$

(iii) Differentiating with respect to 'x' we get,

$$x \cdot \frac{dy}{dx} + y(1) = 0 \qquad \text{[Product rule]}$$

$$\Rightarrow x\frac{dy}{dx} + y = 0$$

which is the require differentiated equation.

(iv) Differentiating with respect to 'x' we get,

$$2x + 2y\frac{dy}{dx} = 0$$

Dividing by 2, we get,

$$x + y\frac{dy}{dx} = 0$$

3. Form the differential equation by eliminating α and β from $(x - \alpha)^2 + (y - \beta)^2 = r^2$.

Sol. $(x - \alpha)^2 + (y - \beta)^2 = r^2$ ... (1)

Differentiating with respect to 'x' we get,

$$2(x - \alpha) + 2(y - \beta)\frac{dy}{dx} = 0 \qquad \text{(or)}$$

$$(x - \alpha) + (y - \beta)\frac{dy}{dx} = 0 \qquad ...(2)$$

Again differentiating with respect to x ,

$$1 + (y - \beta)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$(y - \beta)\frac{d^2y}{dx^2} = -1 - \left(\frac{dy}{dx}\right)^2$$

For convenience we use $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$

The above equation becomes,

$$(y - \beta)\, y'' = -(1 + y'^2)$$

$$y - \beta = -\frac{(1 + y'^2)}{y''}$$

Using (3) in (2) we get,

$$(x - \alpha) - \frac{(1 + y'^2)}{y''}y' = 0$$

(or) $$(x - \alpha) = \frac{1 + y'^2}{y''}y' \qquad ... (4)$$

Now using (3) and (4) in (1) we get

$$\left[\frac{(1 + y'^2)}{y''}y'\right]^2 + \left[\frac{-(1 + y'^2)}{y''}\right]^2 = r^2 \qquad \text{(or)}$$

$$\frac{(1 + y'^2)^2}{(y'')^2}(y'^2 + 1) = r^2$$

$$(1 + y'^2)^3 = r^2\,(y'')^2$$

$$\Rightarrow \quad \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

is the required differential equation.

4. Find the differentiate equation of the family of all straight lines passing through the origin. **[Govt.MQP-2019 ; March - 2020]**

Sol. Let the equation of straight lines passing through the origin be $y = mx$ ... (1)

where m is the arbitrary constant

Differentiating with respect to 'x' we get,

$$\frac{dy}{dx} = m(1) \Rightarrow \frac{dy}{dx} = m \qquad ... (2)$$

Substituting (2) in (1) we get,

$$y = \left(\frac{dy}{dx}\right)x$$

5. Form the differential equation that represents all parabolas each of which has a latus rectum 4*a* and whose axes are parallel to the *x* axis.

Sol. Equation of the family of parabolas with latus rectum 4*a* and whose axes are parallel to the *x*-axis is $(y-k)^2 = 4a(x-h)$

[Where (h, k) is the centre of the parabola]

Differentiating with respect to '*x*' we get,

$$2(y-k)\left(\frac{dy}{dx}\right) = 4a(1) \Rightarrow 2(y-k)\left(\frac{dy}{dx}\right) = 4a \quad ...(1)$$

Differentiating again with respect to *x* we get,

$$2(y-k)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)(2)\frac{dy}{dx} = 0 \quad \text{(Product rule)}$$

$$(y-k)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0 \quad \text{[Divided by 2]}$$

$$y-k = \frac{-\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \quad ...(2)$$

Substituting (2) in (1) we get,

$$2\frac{-\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\left(\frac{dy}{dx}\right) = 4a$$

$$\Rightarrow -2\left(\frac{dy}{dx}\right)^3 = 4a\left(\frac{d^2y}{dx^2}\right)$$

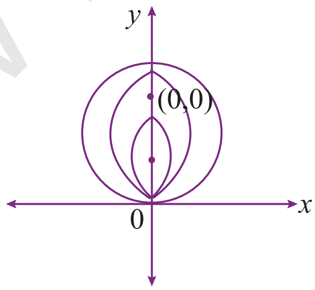
$$\Rightarrow 4a\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right)^3 = 0 \quad \text{[Divided by 2]}$$

$$\Rightarrow 2a\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^3 = 0$$

Which is the required differential equation.

6. Find the differential equation of all circles passing through the origin and having their centers on the *y* axis.

Sol.



The circle pass through the origin. They have their centres at $(0,a)$

The circles have radius *a*. So the equation of the family of circles in given by $x^2 + (y-a)^2 = a^2$

$$x^2 + y^2 - 2ay + a^2 = a^2$$

$$x^2 + y^2 = 2ay \quad ...(1)$$

Differentiating with respect to *x*,

$$2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$$

$$\Rightarrow \quad a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}} \quad ...(2)$$

Using (2) in (1)

$$x^2 + y^2 = 2y\left[\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right]$$

$$x^2\frac{dy}{dx} + y^2\frac{dy}{dx} = 2xy + 2y^2\frac{dy}{dx}$$

$$\Rightarrow (x^2 - y^2)\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

is the required differential equation of all circled passing through origin and having their centers on the *y* - axis.

7. Find the differential equation of the family of parabola with foci at the origin and axis along the *x*-axis.

Sol. Equation of family of parabolas with foci at the origin and axis along the *x*-axis is

$$y^2 = 4a(x+a) \quad ...(1)$$

[∵ the focus is at the origin its vertex will be $(-a, 0)$ and latus rectum is $4a$]

Differentiating with respect to'*x*' we get,

$$2y\left(\frac{dy}{dx}\right) = 4a\,(1)$$

$$\Rightarrow 2y\left(\frac{dy}{dx}\right) = 4a \quad ...(2)$$

$$\text{Also } \frac{2y}{4}\left(\frac{dy}{dx}\right) = a$$

$$\Rightarrow \frac{y}{2}\left(\frac{dy}{dx}\right) = a \quad ...(3)$$

Chapter 5

NUMERICAL METHODS

SNAPSHOT

- Forward difference operator (Δ) = (Delta)
  $\Delta yn = y_{n+1} - y_n$, $n = 0, 1, 2, \ldots$
- $\Delta f(x) = f(x + h) - f(x)$, h is the equal interval of spacing

PROPERTIES OF OPERATOR Δ:

- If c is a constant then $\Delta c = 0$
  Δ is distributive $\Rightarrow \Delta\,(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$
- If c is a constant, then $\Delta c \cdot f(x) = c \cdot \Delta f(x)$.
- If m and n are positive integers then
  $\Delta^m \cdot \Delta^n f(x) = \Delta^{m+n} \cdot f(x)$
- $\Delta\,[f(x) \cdot g(x)] = f(x) \cdot \Delta g(x) + g(x) \cdot \Delta f(x)$
- $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\Delta f(x) - f(x) \cdot \Delta g(x)}{g(x) \cdot g(x+h)}$
- $\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$
- $\Delta^3 y_n = \Delta^2 y_{n+1} - \Delta^2 y_{n,n} = 0, 1, 2 \ldots$
- $\Delta^k y_n = \Delta^{k-1} y_{n+1} - \Delta^{k-1} y_{n,n} = n = 0, 1, 2, \ldots$..

BACKWARD DIFFERENCE OPERATOR ∇ (NEPLA)

- $\nabla y_1 = y_1 - y_0$.
- $\nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1}$, $n = 1, 2, 3 \ldots$
- $\nabla f(x + 2h) = f(x + 2h) - f(x + h)$, h is the interval of spacing.
- $\nabla^n f(x + nh) = \nabla^n f(x)$.

SHIFTING OPERATOR (E)

- $E[f(x_0)] = f(x_0 + h)$
- $E^2 f(x) = E[f(x + h)] = f(x + 2h)$
- $E^n f(x) = f(x + nh)$ and $E^{-n} f(x) = f(x - nh)$.

PROPERTIES OF OPERATOR E

- $E[f_1(x) + f_2(x) + \cdots + f_n(x)] = E f_1(x) + E f_2(x) + \ldots + E f_n(x)$
- $E[c \cdot f(x)] = c \cdot E[f(x)]$ where c is a constant.
- $E^m [E^n f(x)] = E^n [E^m f(x)] = E^{m+n} f(x)$
- If 'n' is a positive integer then,
  $E^n [E^{-n} f(x)] = f(x)$.
- $E(E(y_0)) = E(y_1) = y_2 \Rightarrow E^n y_0 = y_n$.

Relation between the operator Δ, ∇ and E.

- $\Delta = E - 1$
- $E(\Delta f(x)) = \Delta \cdot E f(x)$
- $\nabla = \dfrac{E-1}{E}$
- $(1 + \Delta)(1 - \nabla) = 1$
- $\Delta \nabla = \Delta - \nabla$
- $\nabla = E^{-1} \Delta$.

IMPORTANT FORMULA TO REMEMBER

- $\Delta f(x) = f(x + h) - f(x)$
- $\nabla f(x) = f(x) - f(x - h)$
- $E f(x) = f(x + h)$
- $E^n f(x) = f(x + nh)$

NEWTON'S FORWARD INTERPOLATION FORMULA:

$$y(x = x_0 + nh) = y_0 + \frac{n!}{\Delta y_0} + \frac{n(n-1)}{2!} \Delta^2 y_0 \frac{+n(n-1)(n-2)}{3!} \Delta^3 y_0 + \ldots$$

NEWTONS BACKWARD INTERPOLATION FORMULA:

$$y(x = x_n + nh) = y_n + \frac{n}{1!} \nabla y_n + \frac{n(n+1)}{2!} \nabla^2 y_n + \frac{n(n+1)(n+2)}{3!} \nabla^3 y_n + \ldots v$$

LAGRANGE'S INTERPOLATION FORMULA

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1 + \ldots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n.$$

EXERCISE 5.1

1. Evaluate Δ (log *ax*) [PTA - 6]

Sol. $\Delta(\log ax) = \log a(x+h) - \log ax$

$= \log\left[\frac{a(x+h)}{ax}\right] = \log\left[1+\frac{h}{x}\right]$

2. If $y = x^3 - x^2 + x - 1$, calculate the values of y for x = 0, 1, 2, 3, 4, 5 and form the forward difference table.

Sol. when $x = 0$,
$y = 0+0+0-1 \Rightarrow y = -1$
when $x = 1$,
$y = 1^3 - 1^2 + 1 - 1 \Rightarrow y = 0.$
when $x = 2, y = 2^3 - 2^2 + 2 - 1$
$\Rightarrow y = 8 - 4 + 1 \Rightarrow y = 5$
when $x = 3, y = 3^3 - 3^2 + 3 - 1$
$\Rightarrow y = 27 - 9 + 2 \Rightarrow y = 20$
when $x = 4, y = 4^3 - 4^2 + 4 - 1$
$\Rightarrow y = 64 - 16 + 3 \Rightarrow y = 51$
when $x = 5, y = 5^3 - 5^2 + 5 - 1$
$\Rightarrow y = 125 - 25 + 4 \Rightarrow y = 104$

Hence, the forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	–1					
1	0	1	4	6		
2	5	5	10	6	0	
3	20	15	16	6	0	0
4	51	31	22			
5	104	53				

3. If $h = 1$, then prove that $(E^{-1}\Delta)\, x^3 = 3x^2 - 3x + 1$. [PTA - 4]

Sol. Given $h = 1$

LHS $= (E^{-1}\Delta)\, x^3$
$= \Delta (E^{-1}(x^3))$
$= \Delta (x-h)^3$ $[\because E^{-1} f(x) = f(x - nh)]$
$= \Delta (x-1)^3$ $[\because h = 1]$
$= (x - 1 + 1)^3 - (x-1)^3$ $[\because \Delta f(x) = f(x+h) - f(x)]$
$= x^3 - (x-1)^3$
$= x^3 - (x^3 - 3x^2 + 3x - 1)$ $[\because (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3]$
$= x^3 - x^3 + 3x^2 - 3x + 1$
$= 3x^2 - 3x + 1$
= RHS Hence proved.

4. If $f(x) = x^2 + 3x$, then show that $\Delta f(x) = 2x + 4$

Sol. Given $f(x) = x^2 + 3h$

LHS $= \Delta f(x)$
$= f(x+h) - f(x)$
$= [(x+h)^2 + 3(x+h)] - [x^2 + 3x]$
$= x^2 + h^2 + 2xh + 3x + 3h - x^2 - 3x$
$= h^2 + 2xh + 3h$

when $h = 1$,
LHS $= 1^2 + 2x(1) + 3(1)$
$= 1 + 2x + 3$
$= 2x + 4$ = RHS.

Hence proved.

5. Evaluate $\Delta\left[\frac{1}{(x+1)(x+2)}\right]$ by taking '1' as the interval of differencing. [Aug. - 2021]

Sol. By partial fraction method

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

$$\Rightarrow \frac{1}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$\Rightarrow$ $1 = A(x+2) + B(x+1)$

when $x = -1$, $1 = A[-1+2] \Rightarrow 1 = A$

when $x = -2$, $1 = B[-2+1] \Rightarrow 1 = -B$

$\Rightarrow$ $B = -1.$

$$\therefore \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\therefore \Delta\left[\frac{1}{(x+1)(x+2)}\right] = \Delta\left[\frac{1}{x+1} - \frac{1}{x+2}\right]$$

$$= \left(\frac{1}{x+1+1} - \frac{1}{x+1}\right) - \left(\frac{1}{x+1+2} - \frac{1}{x+2}\right)$$

$[\because \Delta f(x) = f(x+1) - f(x)]$

$$= \left(\frac{1}{x+2} - \frac{1}{x+1}\right) - \left(\frac{1}{x+3} - \frac{1}{x+2}\right) \quad \text{where } h = 1$$

$$= \frac{1}{x+2} - \frac{1}{x+1} - \frac{1}{x+3} + \frac{1}{x+2}$$

$$= \frac{2}{x+2} - \frac{1}{x+1} - \frac{1}{x+3}$$

$$= \frac{2(x+1)(x+3) - 1(x+2)(x+3) - 1(x+1)(x+2)}{(x+1)(x+2)(x+3)}$$

$$= \frac{2(x^2+4x+3)-(x^2+5x+6)-(x^2+3x+2)}{(x+1)(x+2)(x+3)}$$

$$= \frac{2x^2+8x+6-x^2-5x-6-x^2-3x-2}{(x+1)(x+2)(x+3)}$$

$$= \frac{2x^2+8x-8x-2x^2-2}{(x+1)(x+2)(x+3)}$$

$$= \frac{-2}{(x+1)(x+2)(x+3)}$$

$$= \Delta\left[\frac{1}{(x+1)(x+2)}\right] = \frac{-2}{(x+1)(x+2)(x+3)}.$$

6. Find the missing entry in the following table. **[PTA-1; Govt.MQP-2019]**

x	0	1	2	3	4
y	1	3	9	–	81

Sol. Since only four values of $f(x)$ are given, the polynomial which fits the data is of degree 3. Hence fourth differences are zero.

$$\therefore \Delta^4 (y_0) = 0$$

$$\Rightarrow (E-1)^4 (y_0) = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_0 = 0$$

$$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\Rightarrow 81 - 4(y_3) + 6(9) - 4(3) + 1 = 0$$

$$\Rightarrow 81 - 4y_3 + 54 - 12 + 1 = 0$$

$$\Rightarrow 81 + 54 - 11 = 4y_3$$

$$\Rightarrow 124 = 4y_3$$

$$\Rightarrow y_3 = \frac{124}{4} = 31.$$

$$\Rightarrow y_3 = 31$$

7. Following are the population of a district:

year (x)	1881	1891	1901	1911	1921	1931
population (y) in thousands	363	391	421	–	467	501

Find the population of the year 1911. **[PTA-4]**

Sol. Since only five values of $f(x)$ are given, the polynomial which fits the data is of degree 4.

Hence fifth differences are zeros.

$$\therefore \Delta^5 y_0 = 0$$

$$\Rightarrow (E-1)^5\, y_0 = 0$$

$$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_0 = 0$$

$$\Rightarrow y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

$$\Rightarrow 501 - 5(467) + 10y_3 - 10(421) + 5(391) - 363 = 0$$

$$\Rightarrow 501 - 2335 + 10y_3 - 4210 + 1955 - 363 = 0$$

$$\Rightarrow 10y_3 - 4452 = 0$$

$$\Rightarrow 10y_3 = 4452$$

$$\Rightarrow y_3 = \frac{4452}{10} = 445.2$$

Since the population is given in thousands, population of the year 1911

= 445 × 1000 = 4,45,000.

8. Find the missing entries from the following:-

x	0	1	2	3	4	5
$y = f(x)$	0	–	8	15	–	35

Sol. Let the missing entries by y_1 and y_4 **[PTA-3]**

Since only four values of $f(x)$ are given, the polynomial which fits the data is of degree 3.

Hence fourth differences are zero.

$$\therefore \Delta^4 y_k = 0 \Rightarrow (E-1)^4 y_k = 0$$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_k = 0 \quad \text{... (1)}$$

Put $k = 0$ in (1) we get,

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_0 = 0$$

$$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\Rightarrow y_4 - 4(15) + 6(8) - 4(y_1) + 0 = 0$$

$$\Rightarrow y_4 - 60 + 48 - 4y_1 = 0$$

$$\Rightarrow y_4 - 4y_1 - 12 = 0$$

$$\Rightarrow y_4 - 4y_1 = 12 \quad \text{... (2)}$$

put $k = 1$ in (1) we get

$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_1 = 0$$

$$\Rightarrow y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0$$

$$\Rightarrow 35 - 4y_4 + 6(15) - 4(8) + y_1 = 0$$

$$\Rightarrow 35 - 4y_4 + 90 - 32 + y_1 = 0$$

$$\Rightarrow -4y_4 + y_1 = -93 \quad \text{... (3)}$$

$$(2) \times 4 \rightarrow \quad 4y_4 - 16y_1 = 48$$

$$(3) \rightarrow \quad -4y_4 + y_1 = -93$$

$$\text{Adding,} \quad -15y_1 = -45$$

$$\Rightarrow \quad y_1 = \frac{-45}{-15} = 3$$

$$\Rightarrow \quad y_1 = 3$$

Substituting $y_1 = 3$ in (2) we get,

$$y_4 - 4(3) = 12$$

$$y_4 - 12 = 12$$

$$\Rightarrow \quad y_4 = 12 + 12$$

$$\Rightarrow \quad y_4 = 24$$

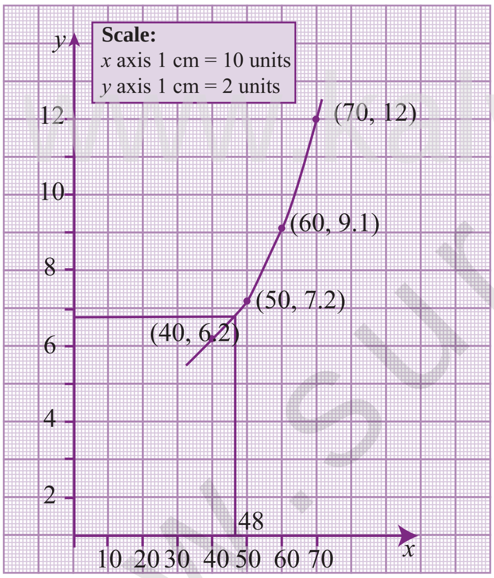
Hence the missing entries are 3 and 24.

EXERCISE 5.2

1. Using graphic method, find the value of y when $x = 48$ from the following data. **[QY-2019]**

x	40	50	60	70
y	6.2	7.2	9.1	12

Sol.



Plot the points (40, 6.2), (50, 7.2), (60, 9.1) and (70, 12). At $x = 48$, draw a vertical line to the graph and from the intersecting point, draw a horizontal line to meet the y-axis

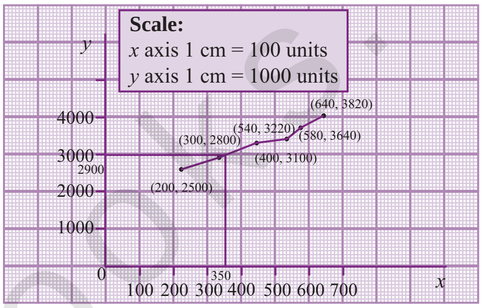
From the graph, we find that when $x = 48$, the value of y is equal to 6.8.

2. The following data relates to indirect labour expenses and the level of output

Months	Jan	Feb	Mar	Apr	May	June
Units of Output	200	300	400	640	540	580
Indirect labour expense (Rs)	2500	2800	3100	3820	3220	3640

Estimate the expenses at a level of output of 350 units, by using graphic method.

Sol. Plot the points (200, 2500), (300, 2800) (400, 3100) (640, 3820), (540, 3220) and (580, 3640).



At $x = 350$, draw a vertical line and from the intersecting point on the curve, draw a horizontal line.

From the graph, we find that when $x = 350$, $y = 2900$.

Hence, the expense at a level of 350 units is ₹ 2900.

3. Using Newton's forward interpolation formula find the cubic polynomial.

x	0	1	2	3
$f(x)$	1	2	1	10

Sol. The forward interpolation formula is

$$y_{(x = x_0 + nh)} = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

Here $x_0 + nh = x$

$$\Rightarrow \quad x_0 = 0, h = 1$$

$$\therefore 0 + n = x$$

$$\Rightarrow \quad n = x.$$

Chapter 7

PROBABILITY DISTRIBUTIONS

MUST KNOW DEFINITIONS

- A random variable X is said to follow Binomial distribution if its probability mass function is given by

$$P(X = x) = P(x) = \begin{cases} {}^nC_x p^x q^{n-x}, & x = 0,1,2,.....n;\ q = 1-p \\ 0, & \text{otherwise} \end{cases}$$

- A random variable X is said to follow Poisson distribution if its probability mass function is given by

$$P(X = x) = \begin{cases} \dfrac{e^{-\lambda}.\lambda^x}{x!}, & x = 0, 1, 2,......\ \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

- A random variable X is said to follow normal distribution if its probability density function is given by

$$f(x\,;\mu\,,5) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \quad -\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0$$

FORMULAE TO REMEMBER

Properties of Binomial Distribution :

- Mean = np and variance = npq
- For binomial distribution, variance < mean
- Binomial distribution is symmetrical if $p = q = 0.5$.
- It is skew symmetric if $p \neq q$.
- It is positively skewed if $p < 0.5$.
- It is negatively skewed if $p > 0.5$.

Properties of Poisson Distribution :

- Mean = Variance = λ

Properties of Normal Distribution :

- The curve is bell shaped and symmetrical about the line $x = \mu$.
- Mean, median and mode of the distribution coincide.
- X - axis is an asymptote to the curve.
- No portion of the curve lies below the X - axis.
- The points of inflexion of the curve are $x = \mu \pm \sigma$.
- The curve is unimodal.
- The max probability occurs at $x = \mu$ and it is $\dfrac{1}{\sigma\sqrt{2\pi}}$.
- $p\,(\mu-\sigma < X < \mu+\sigma) = 0.6826$
- $p\,(\mu-2\sigma < X < \mu+2\sigma) = 0.9544$
- $p\,(\mu-3\sigma < X < \mu+3\sigma) = 0.9973$

Properties of Standard Normal Distribution :

- Its probability density function is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, -\infty < z < \infty$$

- The area under the standard normal curve is 1.
  - 68.26% of area lies between $z = -1$ to $z = 1$.
  - 95.44% of area lies between $z = -2$ and $z = 2$.
  - 99.74% of area lies between $z = -3$ and $z = 3$.

TEXTUAL QUESTIONS

EXERCISE 7.1

1. Define Binomial distribution.

Sol: A random variable X is said to follow binomial distribution with parameter n and p, if it assumes only non-negative value and its probability mass function is given by

$$P(X = x) = p(x) = \begin{cases} {}^nC_x p^x q^{n-x}, & x = 0,1,2,.....n;\ q = 1-p \\ 0, & \text{otherwise} \end{cases}$$

2. Define Bernoulli trials.

Sol: A random experiment whose outcomes are of two types namely success S and failure F, occurring with probabilities p and q is called a Bernoulli trial.

3. Derive the mean and variance of binomial distribution.

Sol: The mean of the binomial distribution

$$E(X) = \sum_{x=0}^{n} x.p(x) = \sum_{x=0}^{n} x.\binom{n}{x} p^x q^{n-x}$$

$$= p.\sum_{x=0}^{n} x.\binom{n}{x} p^{x-1} q^{n-x}$$

[Take p common]

$$= p.\sum_{x=1}^{n} \not{x}.\left(\frac{n}{\not{x}}\right).\binom{n-1}{x-1} p^{x-1} q^{n-x}$$

$$= np.\sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$$

$= np\,(q+p)^{n-1}$ [using binomial theorem]

$(x+a)^n = x^n + {}^nC_1\, x^{n-1} a^1 + ... + a^n$

$= np(1)^{n-1}$ $[\because p+q=1]$

$= np$

$\therefore$ Mean $= E(X) = np$ ... (1)

$$\because {}^nC_x = \frac{n!}{(n-x)!x!} = \frac{n(n-1)!}{(n-x)!\,x(x-1)!} = \frac{n}{x}\left[{}^{n-1}C_{x-1}\right]$$

Now, $E(X^2) = \sum_{x=0}^{n} x^2.\binom{n}{x} p^x q^{n-x}$

$$= \sum_{x=0}^{n} \{x(x-1)+x\}.\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1).\binom{n}{x} p^x q^{n-x} + \sum x.\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1)\frac{n(n-1)}{x(x-1)}\binom{n-2}{n-2} p^{x-2} q^{n-x} + \sum x\binom{n}{x} p^x q^{n-x}$$

$$= n(n-1)p^2\left\{\sum \binom{n-2}{x-2} p^{x-2} q^{n-x}\right\} + np \text{ (using (1))}$$

$= n(n-1)p^2\,(q+p)^{n-2} + np$ [using binomial theory]

$= n(n-1)p^2\,(1) + np$ $\because p+q=1$

$= n(n-1)p^2 + np$ ..... (2)

Variance $= E(X^2) - [E(X)]^2$

$= n(n-1)p^2 + np - (np)^2$ [From (1) & (2)]

$= n^2 \not{p^2} - np^2 + np - n^2 \not{p^2}$

$= np\,(1-p) = npq$ $[\because p+q = 1 \Rightarrow q = 1-p]$

$\therefore$ Mean $= np$ and variance $= npq$.

4. Write down the conditions for which the binomial distribution can be used.

Ans. The Binomial distribution can be used under the following conditions.

(i) The number of trials 'n' is finite.

(ii) The trials are independent of each other.

(iii) The probability of success 'p' is constant for each trial.

(iv) In every trial there are only two possible outcomes namely success or failure.

5. Mention the properties of binomial distribution.

Ans. (i) Binomial distribution is symmetrical if $p = q = 0.5$. It is skew symmetric if $p \neq q$. It is positively skewed if $p < 0.5$ and it is negatively skewed if $p > 0.5$.

(ii) For Binomial distribution, variance is less than mean.

Variance $= npq = (np)q < np$

6. If 5% of the items produced turn out to be defective, then find out the probability that out of 20 items selected at random there are

(i) exactly three defectives [PTA-1; Aug. - 2021]

(ii) atleast two defectives

(iii) exactly 4 defectives

(iv) find the mean and variance

Sol: Given that probability of getting defective item

$$p = 5\% = \frac{5}{100} \Rightarrow q = 1-p = 1-\frac{5}{100} = \frac{95}{100}$$

$n = 20$

$p(x) = {}^nC_x\, p^x\, q^{n-x}, x = 0, 1, 2......n$

(i) P(Exactly 3 defectives)

$= {}^{20}C_3 \left(\frac{5}{100}\right)^3 \left(\frac{95}{100}\right)^{20-3}$

$= {}^{20}C_3\,(0.05)^3\,(0.95)^{17}$

$= \frac{20\times19\times18}{3\times2\times1}(0.05)^3(0.95)^5(0.95)^5(0.95)^5(0.95)^2$

$= (60 \times 19)\,(0.000125)\,(0.7738)\,(0.7738)\,(0.7738)\,(0.9025)$

$= 0.059.$

(ii) P(atleast 2 defectives)

$= P(X \geq 2) = 1-P(X < 2)$

$= 1 - [P(X = 0) + P(X = 1)]$

$= 1 - [{}^{20}C_0\,(0.05)^0\,(0.95)^{20} + {}^{20}C_1(0.05)^1\,(0.95)^{19}]$

$= 1 - [(0.95)^{20} + 20\,(0.05)\,(0.95)^{19}]$

$= 1 - [0.3585 + (0.3774)]$

$= 1 - [0.7359] = 0.2641.$

(iii) P (exactly 4 defectives)

$= P(X = 4) = {}^{20}C_4\,(0.05)^4\,(0.95)^{16}]$

$= \frac{\overset{5}{\cancel{20}}\times19\times\overset{3}{\cancel{\overset{\cancel{6}}{18}}}\times17}{\cancel{4}\times3\times\cancel{2}\times1}(.00000625)\,(0.4402)$

$= (15\times17\times19)\,(0.00000625)\,(0.4402)$

$= 0.0133$

(iv) Find the mean and variance

$$\text{Mean} = np = 20\times\frac{5}{100} = \frac{100}{100} = 1$$

$$\text{Variance} = npq = 20\times\frac{5}{100}\times\frac{95}{100} = 0.95$$

7. In a particular university 40% of the students are having news paper reading habit. Nine university students are selected to find their views on reading habit. Find the probability that

(i) none of those selected have news paper reading habit

(ii) all those selected have news paper reading habit

(iii) atleast two third have news paper reading habit.

Sol: Let the probability of student having reading habit

$p = 40\% = \frac{40}{100} = 0.4$

$\Rightarrow q = 1-p = 1-0.4 = 0.6$

$n = 9$

(i) P (none of those who have selected having reading habit)

$= P(X = 0)$

$= {}^9C_0\,(0.4)^0\,(0.6)^9$ $\left[\because {}^nC_x\, p^x\, q^{n-x} = p(x),\ n = 9, x = 0\right]$

$= (1)\,(1)\,(0.6)^9$

$= (0.6)^9$ $[\because {}^9C_0 = 1]$

$= 0.01008$

(ii) P (all those who have selected have newspaper reading habit)

$= P(X = 9)$ $\left[\because p(x) = {}^nC_x\, p^x\, q^{n-x},\ n = 9, x = 9\right]$

$= {}^9C_9\,(0.4)^9(0.6)^0$

$= (0.4)^9$ $[\because {}^9C_9 = 1]$

$= 0.000261$

(iii) Two thirds of 9 = $\frac{2}{3}\times 9 = 6$

$\therefore$ P(atleast two third have newspaper reading habit)

= P (atleast 6 have newspaper reading habit)

$= P(X \geq 6)$

$= P(X=6)+P(X=7)+P(X=8)+P(X=9)$

$= {}^9C_6\,(0.4)^6\,(0.6)^3 + {}^9C_7\,(0.4)^7\,(0.6)^2 + {}^9C_8\,(0.4)^8\,(0.6)^1 + {}^9C_9\,(0.4)^9\,(0.6)^0$

$= (0.4)^6\,[{}^9C_6\,(0.6)^3 + {}^9C_7(0.4)(0.6)^2 + 9C^8\,(0.4)^2(0.6) + (0.4)^3]$

$= (0.4)^6\,[{}^9C_3(0.216) + {}^9C_2\,(0.144) + {}^9C_1\,(0.096) + 0.64$

$[\because {}^nC_r = {}^nC_{n-r}]$

$$= (0.4)^6 \left[\frac{\not{9}^3 \times \not{8}^4 \times 7}{\not{3} \times \not{2} \times 1}(0.216) + \frac{9 \times \not{8}^4}{\not{2} \times 1}(0.144) + 9(0.096) + .064 \right]$$

$= (0.4)^6$ [18.144 + 5.184 + 0.864 + 0.064]

$= (0.4)^6$ [24.256] = (0.004096) (24.256)

$\therefore$ P (X $\geq$ 6) = 0.09935

8. In a family of 3 children, what is the probability that there will be exactly 2 girls? **[PTA-6]**

Sol: Let p he the probability of getting a girl

$\therefore p = \frac{1}{2}$ [$\because$ one favourable event and total no of events is 2]

$\Rightarrow q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$ and $n = 3$

$\therefore p$ (getting exactly 2 girls)

= P(X = 2)

$= {}^3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1$ [$\because p(x) = {}^nC_x p^x q^{n-x}$, $n = 3$ and $x = 2$]

$= 3\left(\frac{1}{2}\right)^3 = 3 \times \left(\frac{1}{2}\right)^3 = 3 \times \frac{1}{8} = 0.375.$

P (getting exactly 2 girls) = 0.375.

9. Defects in yarn manufactured by a local mill can be approximated by a distribution with a mean of 1.2 defects for every 6 metres of length. If lengths of 6 metres are to be inspected, find the probability of less than 2 defects.

Sol:

Mean = 1.2

np = 1.2, $n = 6$

$6p$ = 1.2

$p = \frac{1.2}{6}$

p = 0.2 $\Rightarrow$ $q = 1 - 0.2 = 0.8$

$p(x) = nc_x p^x q^{n-x}$,

x = 0, 1, 2, ..., 6

$p(x < 2) = p(0) + p(1)$

$= 6c_0 (0.2)^0 + (0.8)^6 + 6c_1 (0.2)^1 + (0.8)^5$

= (1) (1) (0.262144) + 6 (0.2) (0.32768)

= 0.262144 + 0.393216

$p(x < 2)$ = 0.65536

Thus if length of 6 metres are to be inspected, the probability of less than 2 defects is 0.65536.

10. If 18% of the bolts produced by a machine are defective, determine the probability that out of the 4 bolts chosen at random
(i) exactly one will be defective
(ii) none will be defective
(iii) atmost 2 will be defective

Sol: (i) **P (Exactly one defective bolt)**

= P(X = 1) = P= 18% = 0.18

= q = 0.82

$= 4C_1(0.18)^1 (0.82)^3$ [$p(x) = {}^nC_x p^x q^{n-x}$, $n = 4$, $x = 1$]

$= 4(0.18) (0.82)^3$

= (0.72) (0.5513)

P(X = 1) = 0.3969

(i) **P (none will be defective)**

= P(X = 0) [$p(x) = nC_x p^x q^{n-x}$, $n = 4$, $x = 0$]

$= 4C_0 (0.18)^0 (0.82)^4$

$= (0.82)^4$ [$4C_0 = 1$ and $(0.18)^0 = 1$]

P(X = 0) = 0.4521.

(ii) **P (atmost 2 will be defective)**

= P(X $\leq$ 2) = P(X = 0) + P(X = 1) + P(X =2)

$= 0.4521 + 0.3969 + 4C_2(0.18)^2 (0.82)^2$

$= 0.849 + 6(0.18)^2 (0.82)^2$

= 0.849 + 0.1307 = 0.9797.

11. If the probability of success is 0.09, how many trials are needed to have a probability of atleast one success as 1/3 or more ?

Sol: Given probability of success p = 0.09

$\therefore q = 1 - p = 1 - 0.09 = 0.91$

n = 1

Also P(atleast one success) = $\frac{1}{3}$ or more

$\therefore$ P(X $\geq$ 1) = $\frac{1}{3}$

$\Rightarrow$ 1 – P(X < 1) = $\frac{1}{3}$

$\Rightarrow$ P(X < 1) = $1 - \frac{1}{3} = \frac{2}{3}$

$\Rightarrow$ P(X = 0) = $\frac{2}{3}$

$\Rightarrow$ $nC_x p^x q^{n-x} = \frac{2}{3}$

Putting x = 0,

$nC_0 (0.09)^0 (0.91)^{n-0} = \frac{2}{3}$

$\Rightarrow (0.91)^n = \frac{2}{3} = 0.6666$

when (0.91) is multiplied 5 times we are getting 0.6240

$\therefore n$ = 5 or more

Here number of trails are 5 or more.

Chapter 10

OPERATIONS RESEARCH

MUST KNOW DEFINITIONS

Transportation problem	:	The objective of transportation problem is to determine the amount to be transported from each origin to each destinations such that the total transportation cost is minimized.
Feasible solution	:	A feasible solution to a transportation problem is a set of non negative values x_{ij} (i = 1,2 .....m, j = 1, 2, ...n) that satisfies the constraints.
Basic feasible solution	:	If a solution contains not more than $m+n-1$ allocations where m is the number of rows and n is the number of columns then the solution is called basic feasible solutions.
Optimal solution	:	It is a feasible solution which optimizes (minimises) the total transportation cost.
Non degenerate basis feasible solution	:	It is a basic feasible solution contains exactly $m+n-1$ allocations in independent positions.
Degeneracy	:	If a solution contains less than $m+n-1$ allocations, it is called a degenerate basic feasible solution.
Methods of finding basic feasible solutions	:	1) North West Corner rule - (NWC) 2) Least Cost Method - (LCM) 3) Vogel's Approximation Method (VAM)
Assignment Problem	:	To assign the different jobs to different machines (one job per machine) to minimize the overall cost is known as assignment problem.

$$x_{ij} = \begin{cases} 1, \text{ if } i^{th} \text{ job is assigned to } j^{th} \text{ machine} \\ 0, \text{ if } i^{th} \text{ job is not assigned to } j^{th} \text{ machine} \end{cases}$$

LPP form of assignment problem	:	minimize $Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}\, x_{ij}$ subject to the constraints $\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, 3, ... n$ $\sum_{j=1}^{n} x_{ij} = 1, i = 1,2, ... n$ and $x_{ij} = 0$ or 1 for all i, j
Decision making	:	A process of best selection from a self of alternative courses of action, which is supposed to meet objectives up to satisfaction of the decision maker.
Types of decision making	:	(i) Decision making under certainty (ii) Decision making under uncertainty
Maximum criteria	:	Maximizes the minimum possible pay off (pessimistic decision criterion)
Minimax criteria	:	Minimizes the maximum possible pay off

TEXTUAL QUESTIONS

EXERCISE 10.1

1. What is transportation problem?

Ans. A transportation problem is to determine the amount to be transported from each origin to each destinations such that the total transportation cost is minimized.

2. Write mathematical form of transportation problem.

Ans. The objective function is minimize

$Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}\, x_{ij}$ subject to the constraints

$\sum_{j=1}^{n} x_{ij} = a_i$, $i = 1,2, ... m$ (Supply constraints)

$\sum_{i=1}^{m} x_{ij} = b_j$, $j = 1, 2, ...n$ (demand constraints)

$x_{ij} \geq$, 0 for all i, j (non-negative restrictions)

3. What is feasible solution and non degenerate solution in transportation problem?

Ans. **Feasible Solution :** A feasible solution to a transportation problem is a set of non-negative values x_{ij} ($i = 1, 2, ...m, j = 1, 2, ... n$) that satisfies the constraints.

Non - degenerate solution : If a basic feasible solution to a transportation problem contains exactly $m + n - 1$ allocations in independent positions, it is called a non degenerate basic feasible solution.

4. What do you mean by balanced transportation problem?

Ans. If the total supply = total demand, then the given problem is a balanced transportation problem.

5. Find an initial basic feasible solution of the following problem using north west corner rule.

	D_1	D_2	D_3	D_4	Supply
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

Sol : Here, total supply = 19 + 37 + 34 = 90

Total demand = 16 + 18 + 31 + 25 = 90

i.e., Total supply = total demand

∴ The given problem is balanced transportation problem.

∴ We can find an initial basic feasible solution to the given problem.

From the given table, we can choose the cell in the North west Corner. Here, the cell is (O_1, D_1)

x_{11} = min (16, 19) = 16

	D_1	D_2	D_3	D_4	Supply (a_i)
O_1	(16) 5	3	6	2	19/3
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand (b_j)	16/0	18	31	25	

Reduced transportation table is

	D_2	D_3	D_4	Supply
O_1	(3)3	6	2	3/0
O_2	7	9	1	37
O_3	4	7	5	34
	18/15	31	25	

Now the cell in the north west corner is (O_1, D_2)

i.e., x_{12} = min (18, 3) = 3

Reduced transportation table is

	D_2	D_3	D_4	Supply
O_2	(15) 7	9	1	37/22
O_3	4	7	5	34
	15/0	31	25	

Now, the cell in the north west corner is (O_2, D_2)

∴ x_{22} = min (15, 37) = 15

The reduced transportation table is

	D_3	D_4	
O_2	(22) 9	1	22/0
O_3	7	5	34
	31/9	25	

Now the cell in the north west corner is (O_2, D_3)

∴ x_{23} = min (31, 22) = 22

The reduced transportation table is

	D_3	D_4	
O_3	(9) 7	5	34
	9	25	

Now the cell in the north west corner is (O_3, D_3)

∴ x_{33} = min (9, 34) = 9

The reduced transportation table is

(25) 5	34
25	

Thus, we have the following allocations

	D_1	D_2	D_3	D_4	Supply (a_i)
O_1	(16) 5	(3) 3	6	2	19
O_2	4	(15) 7	(22) 9	1	37
O_3	3	4	(9) 7	(25) 5	34
Demand (b_j)	16	18	31	25	

∴Transportation schedule is $O_1 \rightarrow D_1$, $O_1 \rightarrow D_2$, $O_2 \rightarrow D_2$, $O_2 \rightarrow D_3$, $O_3 \rightarrow D_3$, $O_3 \rightarrow D_4$.

Hence, the total transportation cost

= 16 × 5 + 3 × 3 + 15 × 7 +22 × 9+ 9 × 7+ 25 × 5 = 80 + 9 + 105 + 198 + 63 + 125 = ₹ 580

6. Determine an initial basic feasible solution of the following transportation problem by north west corner method

	Bangalore	Nasik	Bhopal	Delhi	Capacity
Chennai	6	8	8	5	30
Madurai	5	11	9	7	40
Trichy	8	9	7	13	50
Demand (Units / day)	35	28	32	25	

Sol :

Given transportation problem is

	Bangalore	Nasik	Bhopal	Delhi	Capacity
Chennai	6	8	8	5	30
Madurai	5	11	9	7	40
Trichy	8	9	7	13	50
Demand (Units / day)	35	28	32	25	

I. allocation

	Bangalore	Nasik	Bhopal	Delhi	Supply
Chennai	(30) 6	8	8	5	30/0
Madurai	5	11	9	7	40
Trichy	8	9	7	13	50
Demand	35/5	28	32	25	

[∵ min (35, 30) = 30]

II. allocation

	Bangalore	Nasik	Bhopal	Delhi	Supply
Madurai	(5) 5	11	9	7	40/35
Trichy	8	9	7	13	50
Demand	5/0	28	32	25	

[∵ min (5, 40) = 5]

III. allocation

	Nasik	Bhopal	Delhi	Supply
Madurai	(28) 11	9	7	35/7
Trichy	9	7	13	50
Demand	28/0	32	25	

[∵ min (28, 35) = 28]

IV. allocation

	Bhopal	Delhi	Supply
Madurai	(7) 9	7	7/0
Trichy	7	13	50
Demand	32/25	25	

[∵ min (32, 7) = 7]

V. allocation

	Bhopal	Delhi	
Trichy	(25) 7	13	50/25
Demand	25/0	25	

[∵ min (25, 50) = 25]

12th STD

Reg. No.

GOVT. SUPPLEMENTARY EXAMINATION August - 2021

Part - III

Business Mathematics and Statistics (with answers)

TIME ALLOWED : 3.00 Hours] [MAXIMUM MARKS : 90

Instructions :

1. Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.
2. Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams

PART - I

Note : (i) Answer all the questions. **[20 × 1 = 20]**

(ii) Choose the most appropriate answer from the given four alternatives and write the option code and the corresponding answer.

1. The rank of the unit matrix of order n is :

(a) $n + 1$ (b) $n - 1$

(c) n^2 (d) n

2. If $T = \begin{matrix} & A & B \\ A & 0\cdot7 & 0\cdot3 \\ B & 0\cdot6 & x \end{matrix}$ is a transition probability matrix, then the value of x is

(a) 0.4 (b) 0.2 (c) 0.7 (d) 0.3

3. $\int \frac{1}{x^3}dx$ is

(a) $\frac{-1}{3x^2}+c$ (b) $\frac{-3}{x^2}+c$

(c) $\frac{-2}{x^2}+c$ (d) $\frac{-1}{2x^2}+c$

4. Area bounded by the curve $y = \frac{1}{x}$ between the limits 1 and 2 is

(a) log 3 sq.units (b) log 2 sq.units

(c) log 4 sq.units (d) log 5 sq.units

5. The demand and supply function of a commodity are $P(x) = (x - 5)^2$ and $S(x) = x^2 + x + 3$ then the equilibrium quantity x_0 is

(a) 3 (b) 5 (c) 1 (d) 2

6. If the marginal revenue of a firm is constant, then the demand function is

(a) C(x) (b) MR (c) AC (d) MC

7. If $y = cx + c - c^3$ then, its differential equation is

(a) $\frac{dy}{dx}+y = \left(\frac{dy}{dx}\right)^3 - x\frac{dy}{dx}$

(b) $y = x\frac{dy}{dx}+\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^3$

(c) $\frac{d^3y}{dx^3} = 0$

(d) $y+\left(\frac{dy}{dx}\right)^3 = x\frac{dy}{dx}-\frac{dy}{dx}$

8. A homogeneous differential equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ can be solved by making substitution,

(a) $x = v\,y$ (b) $y = v\,x$

(c) $x = v$ (d) $v = y\,x$

9. Consumer Price Index are obtained by:

(a) Marshall Edgeworth formula

(b) Paasche's formula

(c) Family budget method formula

(d) Fisher's Ideal formula

10. For the given data, find the value of $\Delta^3 y_0$ is

x	5	6	9	11
y	12	13	15	18

(a) 2 (b) 1 (c) –1 (d) 0

11. Probability which explains x is equal to or less than particular value is classified as

(a) marginal probability

(b) discrete probability

(c) continuous probability

(d) cumulative probability

12. $E[X - E(X)]^2$ is

(a) V(X) (b) E(X)

(c) S.D(X) (d) $E(X^2)$

13. If for a binomial distribution B(n, p) mean = 4 and variance = $\frac{4}{3}$, the probability, $P(X \geq 5)$ is equal to :

(a) $\left(\frac{1}{3}\right)^6$ (b) $\left(\frac{2}{3}\right)^6$

(c) $4\left(\frac{2}{3}\right)^6$ (d) $\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)$

14. Forty percent of the passengers who fly on a certain route do not check in any luggage. The planes on this route seat 15 passengers. For a full flight, what is the mean of the number of passengers who do not check in any luggage?

(a) 7.20 (b) 6.00 (c) 7.50 (d) 6.45

15. A ………… may be finite or infinite according as the number of observation or items in it is finite or infinite.

(a) census (b) parameter

(c) Population (d) none of these

16. "A random sample is a sample selected in such a way that every item in the population has an equal chance of being included" is said by :

(a) Karl Pearson (b) Harper

(c) Dr. Yates (d) Fisher

17. In …………….. the heterogeneous groups are divided into homogeneous groups.

(a) a stratified random sample

(b) systematic random sample

(c) non-probability sample

(d) a simple random sample

18. The seasonal variation means the variations occurring with in

(a) a month

(b) a number of years

(c) a week

(d) a year

19. E ≡

(a) $1 + \nabla$ (b) $1 + \Delta$

(c) $1 - \nabla$ (d) $1 - \Delta$

20. North-West corner refers to______.

(a) bottom right corner

(b) top left corner

(c) bottom left corner

(d) top right corner

PART - II

Note : Answer **any 7** questions. Question number **30** is compulsory. **7 × 2 = 14**

21. Find the rank of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$.

22. The following information is the probability distribution of successes.

No. of Successes X = x	0	1	2
Probability P (x)	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$

Determine the expected number of success.

23. Solve : $\frac{dy}{dx} = ae^y$

24. Find the value of $\Delta \log x$

25. If $P(x) = \begin{cases} \frac{x}{20}, & x = 0, 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$,

(i) $P(X < 3)$ (ii) $P(2 < X \leq 4)$.

26. Integrate $(3 + x)(2 - 5x)$ with respect to x.

27. The mean of a binomial distribution is 5 and standard deviation is 2. Determine the distribution.

28. A sample of 100 students is chosen from a large group of students. The average height of these students is 162 cm and standard deviation (S.D) is 8 cm. Obtain the standard error for the average height of large group of students of 160 cm.

29. State the different methods of measuring trend.

30. Given $U_0 = 5$; $U_1 = 25$, $U_2 = 20$, $U_3 = 15$ and $U_4 = 35$ Find $\Delta^4 U_0$.

PART - III

Note : Answer **any seven** questions. Question number **40** is compulsory. **7 × 3 = 21**

31. Show that the equations $2x + y = 5$, $4x + 2y = 10$ are consistent and solve them.

32. Integrate $\frac{e^{3x} - e^{-3x}}{e^x}$ with respect to x.

33. When the Elasticity function $\frac{E_y}{E_x}$ is $\frac{x}{x-2}$. Find the function when $x = 6$ and $y = 16$.

34. Solve : $ydx - xdy - 3x^2y^2e^{x^3}dx = 0$.

35. Evaluate $\Delta\left[\frac{1}{(x+1)(x+2)}\right]$ by taking '1' as the interval of differencing.