

Exercise – 3.1

1. Simplify each of the following:

(i) $\sqrt[3]{4} \times \sqrt[3]{16}$

(ii) $\frac{\sqrt[4]{1250}}{\sqrt[4]{2}}$

Sol:

(i) $\sqrt[3]{4} \times \sqrt[3]{16}$

$$\Rightarrow \sqrt[3]{4 \times 16} \quad \boxed{\because \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}}$$

$$\Rightarrow \sqrt[3]{64}$$

$$\Rightarrow \sqrt[3]{4^3} \Rightarrow (4^3)^{\frac{1}{3}} \Rightarrow 4^{3 \times \frac{1}{3}} \Rightarrow 4^1 \Rightarrow 4$$

$$\therefore \sqrt[3]{4} \times \sqrt[3]{16} = 4$$

(ii) $\frac{\sqrt[4]{1250}}{\sqrt[4]{2}}$

$$\Rightarrow \sqrt[4]{\frac{1250}{2}} \quad \boxed{\because \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}}$$

$$\Rightarrow \sqrt[4]{\frac{625 \times 2}{2}} \Rightarrow \sqrt[4]{625} \Rightarrow \sqrt[4]{5^4} \Rightarrow (5^4)^{\frac{1}{4}}$$

$$\Rightarrow 5^{4 \times \frac{1}{4}} \Rightarrow 5^1 = 5$$

$$\therefore \sqrt[4]{\frac{1250}{2}} = 5$$

2. Simplify the following expressions:

(i) $(4 + \sqrt{7})(3 + \sqrt{2})$

(ii) $(3 + \sqrt{3})(5 - \sqrt{2})$

(iii) $(\sqrt{5} - 2)(\sqrt{3} - \sqrt{5})$

Sol:

(i) We have

$$\begin{aligned} (4 + \sqrt{7})(3 + \sqrt{2}) &= 4 \times 3 + 4 \times \sqrt{2} + \sqrt{7} \times 3 + \sqrt{7} \times \sqrt{2} \\ &= 12 + 4\sqrt{2} + 3\sqrt{7} + \sqrt{7 \times 2} \end{aligned}$$

$$= 12 + 4\sqrt{2} + 3\sqrt{7} + \sqrt{14}$$

$$\boxed{\because \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}}$$

$$\therefore (4 + \sqrt{7})(3 + \sqrt{2}) = 12 + 4\sqrt{2} + 3\sqrt{7} + \sqrt{14}$$

(ii) We have,

$$(3 + \sqrt{3})(5 - \sqrt{2}) = 3 \times 5 + 3 \times (-\sqrt{2}) + \sqrt{3} \times 5 + \sqrt{3} \times (-\sqrt{2})$$

$$= 15 - 3\sqrt{2} + 5\sqrt{3} - \sqrt{3 \times 2}$$

$$= 15 - 3\sqrt{2} + 5\sqrt{3} - \sqrt{6}$$

$$\boxed{\because \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}}$$

$$\therefore (3 + \sqrt{3})(5 - \sqrt{2}) = 15 - 3\sqrt{2} + 5\sqrt{3} - \sqrt{6}$$

(iii) We have

$$(\sqrt{5} - 2)(\sqrt{3} - \sqrt{5}) = \sqrt{5} \times \sqrt{3} + \sqrt{5} \times (-\sqrt{5}) + (-2) \times \sqrt{3}$$

$$= \sqrt{5 \times 3} - \sqrt{5 \times 5} - 2\sqrt{3} + 2\sqrt{5}$$

$$= \sqrt{15} - \sqrt{5^2} - 2\sqrt{3} + 2\sqrt{5}$$

$$\boxed{\because \sqrt[n]{a} + \sqrt[n]{b} = \sqrt[n]{ab}}$$

$$= \sqrt{15} - \sqrt{5^2} - 2\sqrt{3} + 2\sqrt{5}$$

$$= \sqrt{15} - 5 - 2\sqrt{3} + 2\sqrt{5}$$

$$\therefore (\sqrt{5} - 2)(\sqrt{3} - \sqrt{5}) = \sqrt{15} - 2\sqrt{3} + 2\sqrt{5} - 5$$

3. Simplify the following expressions:

(i) $(11 + \sqrt{11})(11 - \sqrt{11})$

(ii) $(5 + \sqrt{7})(5 - \sqrt{7})$

(iii) $(\sqrt{8} - \sqrt{2})(\sqrt{8} + \sqrt{2})$

(iv) $(3 + \sqrt{3})(3 - \sqrt{3})$

(v) $(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})$

Sol:

(i) We have,

$$(11 + \sqrt{11})(11 - \sqrt{11}) = (11)^2 - (\sqrt{11})^2$$

$$\boxed{\because (a+b)(a-b) = a^2 - b^2}$$

$$= 121 - 11$$

$$= 110$$

$$\therefore (11 + \sqrt{11})(11 - \sqrt{11}) = 110$$

(ii) We have,

$$(5 + \sqrt{7})(5 - \sqrt{7}) = 5^2 - (\sqrt{7})^2$$

$$\boxed{\because (a+b)(a-b) = a^2 - b^2}$$

$$= 25 - 7 = 18$$

$$\therefore (5 + \sqrt{7})(5 - \sqrt{7}) = 18$$

(iii) We have,

$$(\sqrt{8} - \sqrt{2})(\sqrt{8} + \sqrt{2}) = (\sqrt{8})^2 - (\sqrt{2})^2 \quad \boxed{\because (a+b)(a-b) = a^2 - b^2}$$

$$= 8 - 2 = 6$$

$$\therefore (\sqrt{8} - \sqrt{2})(\sqrt{8} + \sqrt{2}) = 6$$

(iv) We have,

$$(3 + \sqrt{3})(3 - \sqrt{3}) = (3)^2 - (\sqrt{3})^2 \quad \boxed{\because (a+b)(a-b) = a^2 - b^2}$$

$$= 9 - 3 = 6$$

$$\therefore (3 + \sqrt{3})(3 - \sqrt{3}) = 6$$

(v) We have,

$$(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2}) = (\sqrt{5})^2 - (\sqrt{2})^2 \quad \boxed{\because (a-b)(a+b) = a^2 - b^2}$$

$$= 5 - 2 = 3$$

$$\therefore (\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2}) = 3$$

4. Simplify the following expressions:

(i) $(\sqrt{3} + \sqrt{7})^2$

(ii) $(\sqrt{5} - \sqrt{3})^2$

(iii) $(2\sqrt{5} + 3\sqrt{2})^2$

Sol:

(i) $(\sqrt{3} + \sqrt{7})^2 = (\sqrt{3})^2 + 2 \times \sqrt{3} \times \sqrt{7} + (\sqrt{7})^2 \quad \boxed{\because (a+b)^2 = a^2 + 2ab + b^2}$

$$= 3 + 2\sqrt{3 \times 7} + 7$$

$$\boxed{\because \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}}$$

$$= 10 + 2\sqrt{21}$$

$$\therefore (\sqrt{3} + \sqrt{7})^2 = 10 + 2\sqrt{21}$$

(ii) We have

$$(\sqrt{5} - \sqrt{3})^2 = (\sqrt{5})^2 - 2 \times \sqrt{5} \times \sqrt{3} + (\sqrt{3})^2$$

$$= 5 - 2\sqrt{5 \times 3} + 3$$

$$\boxed{\because (a-b)^2 = a^2 - 2ab + b^2}$$

$$= 8 - 2\sqrt{15}$$

$$\boxed{\because \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}}$$

$$\therefore (\sqrt{5} - \sqrt{3})^2 = 8 - 2\sqrt{15}$$

(iii) We have

$$(2\sqrt{5} + 3\sqrt{2})^2 = (2\sqrt{5})^2 + 2 \times (2\sqrt{5}) \times (3\sqrt{2}) + (3\sqrt{2})^2 \quad [\because (a+b)^2 = a^2 + 2ab + b^2]$$

$$= 2^2 \times (\sqrt{5})^2 + (2 \times 2 \times 3) \times \sqrt{5 \times 2} + 3^2 (\sqrt{2})^2$$

$$= 4 \times 5 + 12 \times \sqrt{10} + 9 \times 2$$

$$\therefore (ab)^n = a^n \times b^n \text{ and}$$

$$= 20 + 12\sqrt{10} + 18$$

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$$

$$= 38 + 12\sqrt{10}$$

$$\therefore (2\sqrt{5} + 3\sqrt{2})^2 = 38 + 12\sqrt{10}$$

Exercise – 3.2

1. Rationalise the denominator of each of the following (i – vii) :

(i) $\frac{3}{\sqrt{5}}$

(v) $\frac{\sqrt{3} + 1}{\sqrt{2}}$

(ii) $\frac{3}{2\sqrt{5}}$

(vi) $\frac{\sqrt{2} + \sqrt{5}}{\sqrt{3}}$

(iii) $\frac{1}{\sqrt{12}}$

(vii) $\frac{3\sqrt{2}}{\sqrt{5}}$

(iv) $\frac{\sqrt{2}}{\sqrt{5}}$

Sol:

(i) $\frac{3}{5}\sqrt{5}$

(iv) $\frac{1}{5}\sqrt{10}$

(ii) $\frac{3}{10}\sqrt{5}$

(v) $\frac{\sqrt{6} + \sqrt{2}}{2}$

(iii) $\frac{\sqrt{3}}{6}$

(vi) $\frac{\sqrt{6} + \sqrt{5}}{3}$

(vii) $\frac{3\sqrt{10}}{5}$

2. Find the value to three places of decimals of each of the following. It is given that

$$\sqrt{2} = 1.414, \sqrt{3} = 1.732, \sqrt{5} = 2.236 \text{ and } \sqrt{10} = 3.162.$$

(i) $\frac{2}{\sqrt{3}}$

(iv) $\frac{\sqrt{10} + \sqrt{15}}{\sqrt{2}}$

(ii) $\frac{3}{\sqrt{10}}$

(v) $\frac{2 + \sqrt{3}}{3}$

(iii) $\frac{\sqrt{5} + 1}{\sqrt{2}}$

(vi) $\frac{\sqrt{2} - 1}{\sqrt{5}}$

Sol:

Given that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$ and $\sqrt{10} = 3.162$.

(i) We have $\frac{2}{\sqrt{3}}$

Rationalising factor of denominator is $\sqrt{3}$

$$\therefore \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2\sqrt{3}}{3} = \frac{2 \times 1.732}{3} = \frac{3.464}{3}$$

$$= 1.15466667$$

$$= 1.154$$

(ii) We have $\frac{3}{\sqrt{10}}$

Rationalising factor of denominator is $\sqrt{10}$

$$\therefore \frac{3}{\sqrt{10}} = \frac{3}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}} = \frac{3\sqrt{10}}{(\sqrt{10})^2} = \frac{3\sqrt{10}}{10} = \frac{3 \times 3.162}{10} = \frac{9.486}{10}$$

$$= 0.9486$$

$$= 0.948$$

$$\therefore \frac{3}{\sqrt{10}} = 0.948$$

(iii) We have $\frac{\sqrt{5}+1}{\sqrt{2}}$

Rationalising factor of denominator is $\sqrt{2}$.

$$\begin{aligned} \therefore \frac{\sqrt{5}+1}{\sqrt{2}} &= \frac{\sqrt{5}+1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{(\sqrt{5}+1)\sqrt{2}}{(\sqrt{2})^2} \\ &= \frac{\sqrt{5} \times \sqrt{2} + 1 \times \sqrt{2}}{2} \\ &= \frac{\sqrt{5 \times 2} + \sqrt{2}}{2} = \frac{\sqrt{10} + \sqrt{2}}{2} \\ &= \frac{3.162 + 1.414}{2} = \frac{4.576}{2} = 2.288 \end{aligned}$$

$$\therefore \frac{\sqrt{5}+1}{\sqrt{2}} = 2.288$$

(iv) We have $\frac{\sqrt{10} + \sqrt{15}}{\sqrt{2}}$

Rationalising factor of denominator is $\sqrt{2}$

$$\begin{aligned} \therefore \frac{\sqrt{10} + \sqrt{15}}{\sqrt{2}} &= \frac{\sqrt{10} + \sqrt{15}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{(\sqrt{10} + \sqrt{15})\sqrt{2}}{(\sqrt{2})^2} \\ &= \frac{\sqrt{10} \times \sqrt{2} + \sqrt{15} \times \sqrt{2}}{2} \\ &= \frac{\sqrt{10 \times 2} + \sqrt{15 \times 2}}{2} \\ &= \frac{\sqrt{20} + \sqrt{30}}{2} = \frac{\sqrt{2 \times 10} + \sqrt{3 \times 10}}{2} \\ &= \frac{\sqrt{2} \times \sqrt{10} + \sqrt{3} \times \sqrt{10}}{2} \\ &= \frac{(1.414 \times 3.162) + (1.732 \times 3.162)}{2} \\ &= \frac{4.471068 + 5.476584}{2} \end{aligned}$$

$$\Rightarrow \frac{4.471068 + 5.476584}{2}$$

$$\Rightarrow \frac{9.947652}{2} = 4.973826 \approx 4.973$$

(v) We have $\frac{2 + \sqrt{3}}{3}$

$$\Rightarrow \frac{2 + 1.732}{3} = \frac{3.732}{3} = 1.244$$

(vi) We have $\frac{\sqrt{2} - 1}{\sqrt{5}}$

Rationalising factor for $\frac{1}{\sqrt{5}}$ is $\sqrt{5}$

$$\Rightarrow \frac{\sqrt{2} - 1}{\sqrt{5}} = \frac{\sqrt{2} - 1}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{(\sqrt{2} \times \sqrt{5}) - (1 \times \sqrt{5})}{(\sqrt{5})^2}$$

$$\begin{aligned} &= \frac{\sqrt{2 \times 5} - \sqrt{5}}{5} \\ &= \frac{\sqrt{10} - \sqrt{5}}{5} \\ &= \frac{3.162 - 2.236}{5} = \frac{0.926}{5} = 0.1852 \\ &\approx 0.185 \end{aligned}$$

$$\therefore \frac{\sqrt{2}-1}{\sqrt{5}} = 0.185$$

3. Express each one of the following with rational denominator:

(i) $\frac{1}{3+\sqrt{2}}$

(ii) $\frac{1}{\sqrt{6}-\sqrt{5}}$

(iii) $\frac{16}{\sqrt{41}-5}$

(iv) $\frac{30}{5\sqrt{3}-3\sqrt{5}}$

(v) $\frac{1}{2\sqrt{5}-\sqrt{3}}$

(vi) $\frac{\sqrt{3}+1}{2\sqrt{2}-\sqrt{3}}$

(vii) $\frac{6-4\sqrt{2}}{6+4\sqrt{2}}$

(viii) $\frac{3\sqrt{2}+1}{2\sqrt{5}-3}$

(ix) $\frac{b^2}{\sqrt{a^2+b^2}+a}$

Sol:

(i) We have $\frac{1}{3+\sqrt{2}}$

Rationalising factor for $\frac{1}{3+\sqrt{2}}$ is $3-\sqrt{2}$

$$\begin{aligned}\Rightarrow \frac{1}{3+\sqrt{2}} &= \frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} \\ &= \frac{3-\sqrt{2}}{(3)^2 - (\sqrt{2})^2} \quad \boxed{\because (a+b)(a-b) = a^2 - b^2} \\ &= \frac{3-\sqrt{2}}{9-2} = \frac{3-\sqrt{2}}{7} \quad \therefore \frac{1}{3+\sqrt{2}} = \frac{3-\sqrt{2}}{7}\end{aligned}$$

(ii) We have $\frac{1}{\sqrt{6}-\sqrt{5}}$

Rationalising factor for $\frac{1}{\sqrt{6}-\sqrt{5}}$ is $\sqrt{6}+\sqrt{5}$

$$\begin{aligned}\Rightarrow \frac{1}{\sqrt{6}-\sqrt{5}} &= \frac{1}{\sqrt{6}-\sqrt{5}} \times \frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}} \\ &= \frac{\sqrt{6}+\sqrt{5}}{(\sqrt{6}-\sqrt{5})(\sqrt{6}+\sqrt{5})} \\ &= \frac{\sqrt{6}+\sqrt{5}}{(\sqrt{6})^2 - (\sqrt{5})^2} \quad \boxed{\because (a+b)(a-b) = a^2 - b^2} \\ &= \frac{\sqrt{6}+\sqrt{5}}{6-5} \\ &= \frac{\sqrt{6}+\sqrt{5}}{1} = \sqrt{6}+\sqrt{5} \\ \therefore \frac{1}{\sqrt{6}-\sqrt{5}} &= \sqrt{6}+\sqrt{5}\end{aligned}$$

(iii) We have $\frac{16}{\sqrt{41}-5}$

Rationalisation factor for $\frac{1}{\sqrt{41}-5}$ is $(\sqrt{41}+5)$

$$\begin{aligned}\Rightarrow \frac{16}{\sqrt{41}-5} &= \frac{16}{\sqrt{41}-5} \times \frac{\sqrt{41}+5}{\sqrt{41}+5} \\ &= \frac{16(\sqrt{41}+5)}{(\sqrt{41}-5)(\sqrt{41}+5)}\end{aligned}$$

$$= \frac{16(\sqrt{41}+5)}{(\sqrt{41})^2 - (5)^2} \quad \boxed{\because (a-b)(a+b) = a^2 - b^2}$$

$$= \frac{16(\sqrt{41}+5)}{41-25} = \frac{16(\sqrt{41}+5)}{16} = \sqrt{41}+5$$

$$\therefore \frac{16}{41-5} = \sqrt{41}+5$$

(iv) We have $\frac{30}{5\sqrt{3}-3\sqrt{5}}$

Rationalisation factor for $\frac{1}{5\sqrt{3}-3\sqrt{5}}$ is $5\sqrt{3}+3\sqrt{5}$

$$\Rightarrow \frac{30}{5\sqrt{3}-3\sqrt{5}} = \frac{30}{5\sqrt{3}-3\sqrt{5}} \times \frac{5\sqrt{3}+3\sqrt{5}}{5\sqrt{3}+3\sqrt{5}}$$

$$= \frac{30(5\sqrt{3}+3\sqrt{5})}{(5\sqrt{3}-3\sqrt{5})(5\sqrt{3}+3\sqrt{5})}$$

$$= \frac{30(5\sqrt{3}+3\sqrt{5})}{(5\sqrt{3})^2 - (3\sqrt{5})^2} \quad \boxed{\because (a-b)(a+b) = a^2 - b^2}$$

$$= \frac{30(5\sqrt{3}+3\sqrt{5})}{5^2(\sqrt{3})^2 - 3^2(\sqrt{5})^2}$$

$$= \frac{30(5\sqrt{3}+3\sqrt{5})}{25 \times 3 - 9 \times 5} = \frac{30(5\sqrt{3}+3\sqrt{5})}{75-45} = \frac{30(5\sqrt{3}+3\sqrt{5})}{30}$$

$$= 5\sqrt{3}+3\sqrt{5}$$

$$\therefore \frac{30}{5\sqrt{3}-3\sqrt{5}} = 5\sqrt{3}+3\sqrt{5}$$

(v) We have $\frac{1}{2\sqrt{5}-\sqrt{3}}$

Rationalisation factor for $\frac{1}{2\sqrt{5}-\sqrt{3}}$ is $2\sqrt{5}+\sqrt{3}$

$$\Rightarrow \frac{1}{2\sqrt{5}-\sqrt{3}} = \frac{1}{2\sqrt{5}-\sqrt{3}} \times \frac{2\sqrt{5}+\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$$

$$= \frac{2\sqrt{5}+\sqrt{3}}{(2\sqrt{5})^2 - (\sqrt{3})^2}$$

$$\boxed{\because (a-b)(a+b) = a^2 - b^2}$$

$$\begin{aligned}
 &= \frac{2\sqrt{5} + \sqrt{3}}{2^2(\sqrt{5})^2 - (\sqrt{3})^2} \\
 &= \frac{2\sqrt{5} + \sqrt{3}}{4 \times 5 - 3} = \frac{2\sqrt{5} + \sqrt{3}}{20 - 3} = \frac{2\sqrt{5} + \sqrt{3}}{17} \\
 \therefore \frac{1}{2\sqrt{5} - \sqrt{3}} &= \frac{2\sqrt{5} + \sqrt{3}}{17}
 \end{aligned}$$

(vi) We have $\frac{\sqrt{3} + 1}{2\sqrt{2} - \sqrt{3}}$

Rationalisation factor for $\frac{1}{2\sqrt{2} - \sqrt{3}}$ is $2\sqrt{2} + \sqrt{3}$

$$\begin{aligned}
 \Rightarrow \frac{\sqrt{3} + 1}{2\sqrt{2} - \sqrt{3}} &= \frac{\sqrt{3} + 1}{2\sqrt{2} - \sqrt{3}} \times \frac{2\sqrt{2} + \sqrt{3}}{2\sqrt{2} + \sqrt{3}} \\
 &= \frac{(\sqrt{3} + 1)(2\sqrt{2} + \sqrt{3})}{(2\sqrt{2} - \sqrt{3})(2\sqrt{2} + \sqrt{3})} \\
 &= \frac{\sqrt{3} \times 2\sqrt{2} + \sqrt{3} \times \sqrt{3} + 1 \times 2\sqrt{2} + 1 \times \sqrt{3}}{(2\sqrt{2})^2 - (\sqrt{3})^2} \\
 &= \frac{2\sqrt{2} \times 3 + \sqrt{3} \times 3 + 2\sqrt{2} + \sqrt{3}}{2^2(\sqrt{2})^2 - (\sqrt{3})^2} \\
 &= \frac{2\sqrt{6} + 3 + 2\sqrt{2} + \sqrt{3}}{4 \times 2 - 3} = \frac{2\sqrt{6} + 3 + 2\sqrt{2} + \sqrt{3}}{8 - 3} \\
 &= \frac{2\sqrt{6} + 3 + 2\sqrt{2} + \sqrt{3}}{5}
 \end{aligned}$$

$$\therefore (a-b)(a+b) = a^2 - b^2$$

$$\therefore \frac{\sqrt{3} + 1}{2\sqrt{2} - \sqrt{3}} = \frac{2\sqrt{6} + 3 + 2\sqrt{2} + \sqrt{3}}{5}$$

(vii) We have $\frac{6 - 4\sqrt{2}}{6 + 4\sqrt{2}}$

Rationalisation factor for $\frac{1}{6 + 4\sqrt{2}}$ is $6 - 4\sqrt{2}$

$$\begin{aligned}
 \Rightarrow \frac{6 - 4\sqrt{2}}{6 + 4\sqrt{2}} &= \frac{6 - 4\sqrt{2}}{6 + 4\sqrt{2}} \times \frac{6 - 4\sqrt{2}}{6 - 4\sqrt{2}} \\
 &= \frac{(6 - 4\sqrt{2})^2}{(6)^2 - (4\sqrt{2})^2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore (a+b)(a-b) &= a^2 - b^2 \\
 (a-b)(a-b) &= (a-b)^2
 \end{aligned}$$

$$= \frac{6^2 - 2 \times 6 \times 4\sqrt{2} + (4\sqrt{2})^2}{36 - 4^2(\sqrt{2})^2} \quad \boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

$$= \frac{36 - 48\sqrt{2} + 32}{36 - 32}$$

$$= \frac{68 - 48\sqrt{2}}{4} = \frac{4(17 - 12\sqrt{2})}{4} = 17 - 12\sqrt{2}$$

$$\therefore \frac{6 - 4\sqrt{2}}{6 + 4\sqrt{2}} = 17 - 12\sqrt{2}$$

(viii) We have $\frac{3\sqrt{2} + 1}{2\sqrt{5} - 3}$.

Rationalising factor for $\frac{1}{2\sqrt{5} - 3}$ is $2\sqrt{5} + 3$

$$\Rightarrow \frac{3\sqrt{2} + 1}{2\sqrt{5} - 3} = \frac{3\sqrt{2} + 1}{2\sqrt{5} - 3} \times \frac{2\sqrt{5} + 3}{2\sqrt{5} + 3}$$

$$= \frac{(3\sqrt{2} + 1)(2\sqrt{5} + 3)}{(2\sqrt{5} - 3)(2\sqrt{5} + 3)}$$

$$= \frac{3\sqrt{2} \times 2\sqrt{5} + 3\sqrt{2} \times 3 + 1 \times 2\sqrt{5} + 1 \times 3}{(2\sqrt{5})^2 - (3)^2} \quad \boxed{(a-b)(a+b) = a^2 - b^2}$$

$$= \frac{6\sqrt{10} + 9\sqrt{2} + 2\sqrt{5} + 3}{20 - 9} = \frac{6\sqrt{10} + 9\sqrt{2} + 2\sqrt{5} + 3}{11}$$

$$\therefore \frac{3\sqrt{2} + 1}{2\sqrt{5} - 3} = \frac{6\sqrt{10} + 9\sqrt{2} + 2\sqrt{5} + 3}{11}$$

(ix) We have $\frac{b^2}{\sqrt{a^2 + b^2} + a}$

Rationalisation factor for $\frac{1}{\sqrt{a^2 + b^2} + a}$ is $\sqrt{a^2 + b^2} - a$

$$\Rightarrow \frac{b^2}{\sqrt{a^2 + b^2} + a} = \frac{b^2}{\sqrt{a^2 + b^2} + a} \times \frac{\sqrt{a^2 + b^2} - a}{\sqrt{a^2 + b^2} - a}$$

$$= \frac{b^2(\sqrt{a^2 + b^2} - a)}{(\sqrt{a^2 + b^2})^2 - (a)^2} \quad \boxed{(x+y)(x-y) = x^2 - y^2}$$

$$\begin{aligned}
 &= \frac{b^2(\sqrt{a^2+b^2}-a)}{a^2+b^2-a^2} \\
 &= \frac{b^2(\sqrt{a^2+b^2}-a)}{b^2} \\
 &= (\sqrt{a^2+b^2}-a) \\
 \therefore \frac{b^2}{\sqrt{a^2+b^2}+a} &= \sqrt{a^2+b^2}-a
 \end{aligned}$$

4. Rationalize the denominator and simplify:

(i) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

(ii) $\frac{5+2\sqrt{3}}{7+4\sqrt{3}}$

(iii) $\frac{1+\sqrt{2}}{3-2\sqrt{2}}$

(iv) $\frac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}}$

(v) $\frac{4\sqrt{3}+5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$

(vi) $\frac{2\sqrt{3}-\sqrt{5}}{2\sqrt{2}+3\sqrt{3}}$

Sol:

(i) We have $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

Rationalisation factor for $\frac{1}{\sqrt{a}+\sqrt{b}}$ is $\sqrt{a}-\sqrt{b}$

\Rightarrow for $\frac{1}{\sqrt{3}+\sqrt{2}}$ it is $\sqrt{3}-\sqrt{2}$

$$\therefore \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3})^2+(\sqrt{2})^2}$$

$\begin{aligned} \therefore (a-b)(a-b) &= (a-b)^2 \\ \text{and } (a+b)(a-b) &= a^2-b^2 \end{aligned}$

$$\begin{aligned}
 &= \frac{(\sqrt{3})^2 - 2 \times \sqrt{3} \times \sqrt{2} + (\sqrt{2})^2}{3-2} && \boxed{\because (a-b)^2 = a^2 - 2ab + b^2} \\
 &= \frac{3 - 2\sqrt{6} + 2}{1} = 5 - 2\sqrt{6} \\
 \therefore \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}} &= 5 - 2\sqrt{6}
 \end{aligned}$$

(ii) We have $\frac{5+2\sqrt{3}}{7+4\sqrt{3}}$

Rationalising factor for $\frac{1}{a+b\sqrt{c}}$ is $a-b\sqrt{c}$

$$\Rightarrow \text{for } \frac{1}{7+4\sqrt{3}} \text{ is } 7-4\sqrt{3}$$

$$\begin{aligned}
 \therefore \frac{5+2\sqrt{3}}{7+4\sqrt{3}} &= \frac{5+2\sqrt{3}}{7+4\sqrt{3}} \times \frac{7-4\sqrt{3}}{7-4\sqrt{3}} \\
 &= \frac{5 \times 7 + 5 \times (-4\sqrt{3}) + 2\sqrt{3} \times 7 + 2\sqrt{3} \times (-4\sqrt{3})}{7^2 - (4\sqrt{3})^2} && \boxed{\because (a+b)(a-b) = a^2 - b^2}
 \end{aligned}$$

$$= \frac{35 - 20\sqrt{3} + 14\sqrt{3} - 8 \times 3}{49 - 48}$$

$$= \frac{35 - 24 - 6\sqrt{3}}{1} = 11 - 6\sqrt{3}$$

$$\therefore \frac{5+2\sqrt{3}}{7+4\sqrt{3}} = 11 - 6\sqrt{3}$$

(iii) We have $\frac{1+\sqrt{2}}{3-2\sqrt{2}}$

Rationalisation factor for $\frac{1}{a-b\sqrt{c}}$ is $a+b\sqrt{c}$

$$\Rightarrow \text{for } \frac{1}{3-2\sqrt{2}} \text{ is } 3+2\sqrt{2}$$

$$\begin{aligned}
 \therefore \frac{1+\sqrt{2}}{3-2\sqrt{2}} &= \frac{1+\sqrt{2}}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}} \\
 &= \frac{1 \times 3 + 1 \times 2\sqrt{2} + \sqrt{2} \times 3 + \sqrt{2} \times 2\sqrt{2}}{3^2 - (2\sqrt{2})^2} && \boxed{\because (a-b)(a+b) = a^2 - b^2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{3+2\sqrt{2}+3\sqrt{2}+2\times 2}{9-8} \\
 &= \frac{3+4+5\sqrt{2}}{1} = \frac{7+5\sqrt{2}}{1} = 7+5\sqrt{2}
 \end{aligned}$$

$$\therefore \frac{1+\sqrt{2}}{3-2\sqrt{2}} = 7+5\sqrt{2}$$

(iv) We have $\frac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}}$

Rationalisation factor for $\frac{1}{a\sqrt{b}-c\sqrt{d}}$ is $a\sqrt{b}+c\sqrt{d}$

$$\Rightarrow \text{for } \frac{1}{3\sqrt{5}-2\sqrt{6}} \text{ is } 3\sqrt{5}+2\sqrt{6}$$

$$\begin{aligned}
 \therefore \frac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}} &= \frac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}} \times \frac{3\sqrt{5}+2\sqrt{6}}{3\sqrt{5}+2\sqrt{6}} \\
 &= \frac{2\sqrt{6} \times 3\sqrt{5} + 2\sqrt{6} \times 2\sqrt{6} + (-\sqrt{5})(3\sqrt{5}) + (-\sqrt{5})(2\sqrt{6})}{(3\sqrt{5})^2 - (2\sqrt{6})^2}
 \end{aligned}$$

$$\therefore (a-b)(a+b) = a^2 - b^2$$

$$= \frac{6\sqrt{30} + 4 \times 6 - 3 \times 5 - 2\sqrt{5} \times 6}{45 - 24}$$

$$= \frac{6\sqrt{30} + 24 - 15 - 2\sqrt{30}}{21}$$

$$= \frac{9 + 4\sqrt{30}}{21}$$

$$\therefore \frac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}} = \frac{9+4\sqrt{30}}{21}$$

(v) We have $\frac{4\sqrt{3}+5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$

Rationalisation factor for $\frac{1}{\sqrt{a}+\sqrt{b}}$ is $\sqrt{a}-\sqrt{b}$

$$\Rightarrow \text{for } \frac{1}{\sqrt{48}+\sqrt{18}} \text{ is } \sqrt{48}-\sqrt{18}$$

$$\therefore \frac{4\sqrt{3}+5\sqrt{2}}{\sqrt{48}+\sqrt{18}} = \frac{4\sqrt{3}+5\sqrt{2}}{\sqrt{48}+\sqrt{18}} \times \frac{\sqrt{48}-\sqrt{18}}{\sqrt{48}-\sqrt{18}}$$

$$= \frac{4\sqrt{3} \times \sqrt{48} + 4\sqrt{3} \times (-\sqrt{18}) + 5\sqrt{2} \times \sqrt{48} + 5\sqrt{2} \times (-\sqrt{18})}{(\sqrt{48})^2 - (\sqrt{18})^2} \quad \because (a+b)(a-b) = a^2 - b^2$$

$$= \frac{4\sqrt{3} \times \sqrt{3 \times 16} - 4\sqrt{3} \times \sqrt{2 \times 9} + 5\sqrt{2} \times \sqrt{3 \times 16} - 5\sqrt{2} \times \sqrt{2 \times 9}}{48 - 18}$$

$$= \frac{4\sqrt{3} \times 4\sqrt{3} - 4\sqrt{3} \times 3\sqrt{2} + 5\sqrt{2} \times 4\sqrt{3} - 5\sqrt{2} \times 3\sqrt{2}}{30}$$

$$= \frac{48 - 12\sqrt{6} + 20\sqrt{6} - 30}{30}$$

$$= \frac{18 + 8\sqrt{6}}{30} = \frac{2(9 + 4\sqrt{6})}{30} = \frac{9 + 4\sqrt{6}}{15}$$

$$\therefore \frac{4\sqrt{3} + 5\sqrt{2}}{\sqrt{48} + \sqrt{18}} = \frac{9 + 4\sqrt{6}}{\sqrt{15}}$$

(vi) We have $\frac{2\sqrt{3} - \sqrt{5}}{2\sqrt{2} + 3\sqrt{3}}$

Rationalisation factor for $\frac{1}{a\sqrt{b} + c\sqrt{d}}$ is $a\sqrt{b} - c\sqrt{d}$

$$\Rightarrow \text{for } \frac{1}{2\sqrt{2} + 3\sqrt{3}} \text{ is } 2\sqrt{2} - 3\sqrt{3}$$

$$\therefore \frac{2\sqrt{3} - \sqrt{5}}{2\sqrt{2} + 3\sqrt{3}} = \frac{2\sqrt{3} - \sqrt{5}}{2\sqrt{2} + 3\sqrt{3}} \times \frac{2\sqrt{2} - 3\sqrt{3}}{2\sqrt{2} - 3\sqrt{3}}$$

$$= \frac{2\sqrt{3} \times 2\sqrt{2} + 2\sqrt{3} \times (-3\sqrt{3}) + (-\sqrt{5})(2\sqrt{2}) + (-\sqrt{5})(-3\sqrt{3})}{(2\sqrt{2})^2 - (3\sqrt{3})^2} \quad \because (a+b)(a-b) = a^2 - b^2$$

$$= \frac{4\sqrt{6} - 6\sqrt{3^2} - 2\sqrt{10} + 3\sqrt{15}}{8 - 27}$$

$$= \frac{4\sqrt{6} - 18 - 2\sqrt{10} + 3\sqrt{15}}{-19}$$

$$= \frac{-(18 - 3\sqrt{15} + 2\sqrt{10} - 4\sqrt{6})}{-19} = \frac{18 - 3\sqrt{15} + 2\sqrt{10} - 4\sqrt{6}}{19}$$

$$\therefore \frac{2\sqrt{3} - \sqrt{5}}{2\sqrt{2} + 3\sqrt{3}} = \frac{18 - 3\sqrt{15} + 2\sqrt{10} - 4\sqrt{6}}{19}$$

5. Simplify:

$$(i) \quad \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} + \frac{\sqrt{12}}{\sqrt{3}-\sqrt{2}}$$

$$(ii) \quad \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} + \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$(iii) \quad \frac{7+3\sqrt{5}}{3+\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}}$$

$$(iv) \quad \frac{1}{2+\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}} + \frac{1}{2-\sqrt{5}}$$

$$(v) \quad \frac{2}{\sqrt{5}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{2}} - \frac{3}{\sqrt{5}+\sqrt{2}}$$

Sol:

$$(i) \text{ We have } \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} + \frac{\sqrt{12}}{\sqrt{3}-\sqrt{2}}$$

Rationalisation factor for $\frac{1}{3\sqrt{2}+2\sqrt{3}}$ is $3\sqrt{2}-2\sqrt{3}$ and for $\frac{1}{\sqrt{3}-\sqrt{2}}$ is $\sqrt{3}+\sqrt{2}$

$$\Rightarrow \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} \times \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}} + \frac{\sqrt{4 \times 3}}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$\Rightarrow \frac{(3\sqrt{2}-2\sqrt{3})^2}{(3\sqrt{2})^2 - (2\sqrt{3})^2} + \frac{(\sqrt{4 \times 3})(\sqrt{3}+\sqrt{2})}{(\sqrt{3})^2 - (\sqrt{2})^2}$$

$$\begin{array}{l} \because (a-b)(a-b) = (a-b)^2 \\ (a+b)(a-b) = a^2 - b^2 \end{array}$$

$$\Rightarrow \frac{(3\sqrt{2})^2 - 2 \times 3\sqrt{2} \times 2\sqrt{3} + (2\sqrt{3})^2}{3^2(\sqrt{2})^2 - (2)^2(\sqrt{3})^2} + \frac{2\sqrt{3} \times \sqrt{3} + 2\sqrt{3} \times \sqrt{2}}{3-2}$$

$$\Rightarrow \frac{18-12\sqrt{6}+12}{18-12} + \frac{6+2\sqrt{6}}{1}$$

$$\Rightarrow \frac{30-12\sqrt{6}}{6} + 6+2\sqrt{6}$$

$$\Rightarrow \frac{6(5-2\sqrt{6})}{6} + 6+2\sqrt{6}$$

$$\Rightarrow 5-2\sqrt{6}+6+2\sqrt{6} \Rightarrow 5+6 \Rightarrow 11$$

$$\therefore \frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} + \frac{\sqrt{12}}{\sqrt{3}-\sqrt{2}} = 11$$

$$(ii) \text{ we have } \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} + \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

Rationalisation factor for $\frac{1}{\sqrt{a} \pm \sqrt{b}}$ is $\sqrt{a} \mp \sqrt{b}$

\Rightarrow for $\frac{1}{\sqrt{5} - \sqrt{3}}$ it is $\sqrt{5} + \sqrt{3}$ and for $\frac{1}{\sqrt{5} + \sqrt{3}}$ it is $\sqrt{5} - \sqrt{3}$

$$\begin{aligned} &\Rightarrow \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} + \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} \times \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}} \\ &\Rightarrow \frac{(\sqrt{5} + \sqrt{3})(\sqrt{5} + \sqrt{3})}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} + \frac{(\sqrt{5} - \sqrt{3})(\sqrt{5} - \sqrt{3})}{(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})} \end{aligned}$$

$$\Rightarrow \frac{(\sqrt{5} + \sqrt{3})^2}{(\sqrt{5})^2 - (\sqrt{3})^2} + \frac{(\sqrt{5} - \sqrt{3})^2}{(\sqrt{5})^2 - (\sqrt{3})^2}$$

$$\begin{aligned} \because (a+b)(a+b) &= (a+b)^2 \\ (a-b)(a-b) &= (a-b)^2 \\ (a+b)(a-b) &= a^2 - b^2 \end{aligned}$$

$$\Rightarrow \frac{(\sqrt{5})^2 + 2\sqrt{5}\sqrt{3} + (\sqrt{3})^2}{5-3} + \frac{(\sqrt{5})^2 - 2\sqrt{5}\sqrt{3} + (\sqrt{3})^2}{5-3}$$

$$\Rightarrow \frac{5 + 2\sqrt{15} + 3}{2} + \frac{5 - 2\sqrt{15} + 3}{2}$$

$$\Rightarrow \frac{8 + 2\sqrt{15}}{2} + \frac{8 - 2\sqrt{15}}{2}$$

$$\Rightarrow \frac{2(4 + \sqrt{15})}{2} + \frac{2(4 - \sqrt{15})}{2}$$

$$\Rightarrow 4 + \sqrt{15} + 4 - \sqrt{15}$$

$$\Rightarrow 4 + 4 \Rightarrow 8$$

$$\therefore \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} + \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} = 8$$

(iii) We have $\frac{7+3\sqrt{5}}{3+\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}}$

Rationalisation factor for $\frac{1}{a \pm \sqrt{b}}$ is $a \mp \sqrt{b}$

\Rightarrow for $\frac{1}{3+\sqrt{5}}$ it is $3-\sqrt{5}$ and for $\frac{1}{3-\sqrt{5}}$ it is $3+\sqrt{5}$

$$\Rightarrow \frac{7+3\sqrt{5}}{3+\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}}$$

$$\Rightarrow \frac{7+3\sqrt{5}}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$$

$$\begin{aligned} &\Rightarrow \frac{(7+3\sqrt{5})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} - \frac{(7-3\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} \\ &\Rightarrow \frac{7 \times 3 + 7 \times (-\sqrt{5}) + 3\sqrt{5} \times 3 + 3\sqrt{5} \times (-\sqrt{5})}{3^2 - (\sqrt{5})^2} \\ &\quad - \frac{7 \times 3 + 7 \times \sqrt{5} + (-3\sqrt{5}) \times 3 + (-3\sqrt{5}) \times \sqrt{5}}{3^2 - (\sqrt{5})^2} \quad \boxed{\because (a-b)(a+b) = a^2 - b^2} \\ &\Rightarrow \frac{21 - 7\sqrt{5} + 9\sqrt{5} - 3 \times 5}{9 - 5} - \frac{21 + 7\sqrt{5} - 9\sqrt{5} - 3 \times 5}{9 - 5} \\ &\Rightarrow \frac{21 - 15 + 2\sqrt{5}}{4} - \frac{21 - 15 + 2\sqrt{5}}{4} \\ &\Rightarrow \frac{6 + 2\sqrt{5}}{4} - \frac{6 - 2\sqrt{5}}{4} \\ &\Rightarrow \frac{6 + 2\sqrt{5} - (6 - 2\sqrt{5})}{4} \Rightarrow \frac{6 + 2\sqrt{5} - 6 + 2\sqrt{5}}{4} \Rightarrow \frac{4\sqrt{5}}{4} \Rightarrow \sqrt{5} \\ &\therefore \frac{7+3\sqrt{5}}{3+\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}} = \sqrt{5} \end{aligned}$$

(iv) We have $\frac{1}{2+\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}} + \frac{1}{2-\sqrt{5}}$

Rationalisation factor for $\frac{1}{a \pm \sqrt{b}}$ is $a \mp \sqrt{b}$

$$\Rightarrow \text{for } \frac{1}{2+\sqrt{3}} \text{ it is } 2-\sqrt{3} \text{ and for } \frac{1}{2-\sqrt{5}} \text{ it is } 2+\sqrt{5}$$

And also, rationalisation factor for $\frac{1}{\sqrt{a} \pm \sqrt{b}}$ is $\sqrt{a} \mp \sqrt{b} \Rightarrow$ for $\frac{1}{\sqrt{5}-\sqrt{3}}$ it is $\sqrt{5} + \sqrt{3}$

$$\begin{aligned} &\Rightarrow \frac{1}{2+\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}} + \frac{1}{2-\sqrt{5}} \\ &\Rightarrow \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} + \frac{1}{2-\sqrt{5}} \times \frac{2+\sqrt{5}}{2+\sqrt{5}} \\ &\Rightarrow \frac{2-\sqrt{3}}{2^2 - (\sqrt{3})^2} + \frac{2(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} + \frac{2+\sqrt{5}}{2^2 - (\sqrt{5})^2} \quad \boxed{\because (a+b)(a-b) = a^2 - b^2} \\ &\Rightarrow \frac{2-\sqrt{3}}{4-3} + \frac{2\sqrt{5}+2\sqrt{3}}{5-3} + \frac{2+\sqrt{5}}{4-5} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{2-\sqrt{3}}{1} + \frac{2\sqrt{5}+2\sqrt{3}}{2} + \frac{2+\sqrt{5}}{-1} \\
&\Rightarrow 2-\sqrt{3} + \frac{2(\sqrt{5}+\sqrt{3})}{2} - (2+\sqrt{5}) \\
&\Rightarrow 2-\sqrt{3} + \sqrt{5} + \sqrt{3} - 2 - \sqrt{5} \\
&\Rightarrow 0 \\
&\therefore \frac{1}{2+\sqrt{3}} + \frac{2}{\sqrt{5}-\sqrt{3}} + \frac{1}{2-\sqrt{5}} = 0
\end{aligned}$$

(v) We have,

$$\begin{aligned}
&\frac{2}{\sqrt{5}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{2}} - \frac{3}{\sqrt{5}+\sqrt{2}} \\
&\text{Rationalisation factor for } \frac{1}{\sqrt{a} \pm \sqrt{b}} \text{ is } \sqrt{a} \mp \sqrt{b} \Rightarrow \text{for } \frac{1}{\sqrt{5}+\sqrt{3}} \text{ it is } \sqrt{5}-\sqrt{3} \\
&\Rightarrow \text{for } \frac{1}{\sqrt{3}+\sqrt{2}} \text{ it is } \sqrt{3}-\sqrt{2} \Rightarrow \text{for } \frac{1}{\sqrt{5}+\sqrt{2}} \text{ it is } \sqrt{5}-\sqrt{2} \\
&\Rightarrow \frac{2}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{3}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} \\
&\Rightarrow \frac{2(\sqrt{5}-\sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} + \frac{\sqrt{3}-\sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} - \frac{3(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2 - (\sqrt{2})^2} \\
&\Rightarrow \frac{2(\sqrt{5}-\sqrt{3})}{5-3} + \frac{\sqrt{3}-\sqrt{2}}{3-2} - \frac{3(\sqrt{5}-\sqrt{2})}{5-2} \\
&\Rightarrow \frac{2(\sqrt{5}-\sqrt{3})}{2} + \frac{\sqrt{3}-\sqrt{2}}{1} - \frac{3(\sqrt{5}-\sqrt{2})}{3} \\
&\Rightarrow \sqrt{5}-\sqrt{3} + \sqrt{3}-\sqrt{2} - \sqrt{5} + \sqrt{2} \\
&\Rightarrow 0 \\
&\therefore \frac{2}{5+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{2}} - \frac{3}{\sqrt{5}+\sqrt{2}} = 0
\end{aligned}$$

6. In each of the following determine rational numbers a and b:

$$(i) \quad \frac{\sqrt{3}-1}{\sqrt{3}+1} = a-b\sqrt{3}$$

$$(ii) \quad \frac{4+\sqrt{2}}{2+\sqrt{2}} = a-\sqrt{b}$$

$$(iii) \quad \frac{3+\sqrt{2}}{3-\sqrt{2}} = a+b\sqrt{2}$$

$$(iv) \quad \frac{5+3\sqrt{3}}{7+4\sqrt{3}} = a+b\sqrt{3}$$

$$(v) \quad \frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} = a-b\sqrt{77}$$

$$(vi) \quad \frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a+b\sqrt{5}$$

Sol:

$$(i) \text{ Given } \frac{\sqrt{3}-1}{\sqrt{3}+1} = a-b\sqrt{3}$$

Rationalisation factor for $\frac{1}{\sqrt{x+y}}$ is $\sqrt{x-y} \Rightarrow$ for $\frac{1}{\sqrt{3}+1}$ is $\sqrt{3}-1$

$$\Rightarrow \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - (1)^2}$$

$$\begin{array}{l} \because (a-b)(a-b) = (a-b)^2 \\ (a+b)(a-b) = a^2 - b^2 \end{array}$$

$$= \frac{(\sqrt{3})^2 - 2\sqrt{3} \times 1 + (1)^2}{3-1}$$

$$\because (a-b)^2 = a^2 - 2ab + b^2$$

$$= \frac{3-2\sqrt{3}+1}{2}$$

$$= \frac{4-2\sqrt{3}}{2} = \frac{2(2-\sqrt{3})}{2} = 2-\sqrt{3}$$

We have

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} = a-b\sqrt{3}$$

$$\Rightarrow 2-\sqrt{3} = a-b\sqrt{3} \Rightarrow 2-(1)\sqrt{3} = a-b\sqrt{3}$$

On equating rational and irrational parts, we get $a = 2$ and $b = 1$

(ii) Given that

$$\frac{4+\sqrt{2}}{2+\sqrt{2}} = a - \sqrt{b}$$

Rationalisation factor for $\frac{1}{a \mp \sqrt{b}}$ is $a \mp \sqrt{b} \Rightarrow$ for $\frac{1}{2+\sqrt{2}}$ it is $2-\sqrt{2}$

$$\Rightarrow \frac{4+\sqrt{2}}{2+\sqrt{2}} = \frac{4+\sqrt{2}}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}}$$

$$= \frac{4 \times 2 + \sqrt{2} \times 2 + 4 \times (-\sqrt{2}) + \sqrt{2} \times (-\sqrt{2})}{2^2 - (\sqrt{2})^2}$$

$$\boxed{\because (a+b)(a-b) = a^2 - b^2}$$

$$= \frac{8 + 2\sqrt{2} - 4\sqrt{2} - 2}{4 - 2}$$

$$= \frac{6 - 2\sqrt{2}}{2} = \frac{2(3 - \sqrt{2})}{2} = 3 - \sqrt{2}$$

We have,

$$\frac{4+\sqrt{2}}{2+\sqrt{2}} = a - \sqrt{b}$$

$$\Rightarrow 3 - \sqrt{2} = a - \sqrt{b}$$

On equating rational and irrational parts

We get

$$a = 3 \text{ and } b = 2$$

(iii) Given that

$$\frac{3+\sqrt{2}}{3-\sqrt{2}} = a + b\sqrt{2}$$

The rationalisation factor for $\frac{1}{a - \sqrt{b}}$ is $a + \sqrt{b} \Rightarrow$ for $\frac{1}{3-\sqrt{2}}$ it is $3+\sqrt{2}$

$$\Rightarrow \frac{3+\sqrt{2}}{3-\sqrt{2}} = \frac{3+\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$

$$= \frac{(3+\sqrt{2})^2}{3^2 - (\sqrt{2})^2}$$

$$\boxed{\begin{aligned} \because (a+b)(a+b) &= (a+b)^2 \\ (a+b)(a-b) &= a^2 - b^2 \end{aligned}}$$

$$= \frac{3^2 + 2 \times 3\sqrt{2} + (\sqrt{2})^2}{9 - 2}$$

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

$$= \frac{9 + 6\sqrt{2} + 2}{7} = \frac{11 + 6\sqrt{2}}{7} = \frac{11}{7} + \frac{6}{7}\sqrt{2}$$

We have

$$\frac{3+\sqrt{2}}{3-\sqrt{2}} = a+b\sqrt{2}$$

$$\Rightarrow \frac{11}{7} + \frac{6}{7}\sqrt{2} = a+b\sqrt{2}$$

On equating rational and irrational parts

We get

$$a = \frac{11}{7} \text{ and } b = \frac{6}{7}$$

(iv) Given that

$$\frac{5+3\sqrt{3}}{7+4\sqrt{3}} = a+b\sqrt{3}$$

Rationalisation factor for $\frac{1}{a+b\sqrt{c}}$ is $a-b\sqrt{c} \Rightarrow$ for $\frac{1}{7+4\sqrt{3}}$ it is $7-4\sqrt{3}$

$$\Rightarrow \frac{5+3\sqrt{3}}{7+4\sqrt{3}} = \frac{5+3\sqrt{3}}{7+4\sqrt{3}} \times \frac{7-4\sqrt{3}}{7-4\sqrt{3}}$$

$$= \frac{5 \times 7 + 5 \times (-4\sqrt{3}) + 3\sqrt{3} \times 7 + 3\sqrt{3}(-4\sqrt{3})}{(7)^2 - (4\sqrt{3})^2}$$

$$\boxed{\because (a+b)(a-b) = a^2 - b^2}$$

$$= \frac{35 - 20\sqrt{3} + 21\sqrt{3} - 12 \times 3}{49 - 48}$$

$$= \frac{35 - 36 + \sqrt{3}}{1} = \frac{\sqrt{3} - 1}{1} = \sqrt{3} - 1$$

We have

$$\frac{5+3\sqrt{3}}{7+4\sqrt{3}} = a+b\sqrt{3}$$

$$\Rightarrow \sqrt{3} - 1 = a+b\sqrt{3}$$

$$\Rightarrow -1 + (1)\sqrt{3} = a+b\sqrt{3}$$

On equating the rational and irrational parts

We get

$$a = -1 \text{ and } b = 1$$

(v) Given that,

$$\frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} = a-b\sqrt{7}$$

We know that rationalisation factor for $\frac{1}{\sqrt{a}+\sqrt{b}}$ is $\sqrt{a}-\sqrt{b} \Rightarrow$ for $\frac{1}{\sqrt{11}+\sqrt{7}}$ it is

$$\sqrt{11}-\sqrt{7}$$

$$\begin{aligned} \Rightarrow \frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} &= \frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} \times \frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}-\sqrt{7}} \\ &= \frac{(\sqrt{11}-\sqrt{7})^2}{(\sqrt{11})^2+(\sqrt{7})^2} && \boxed{\begin{aligned} \because (a-b)(a-b) &= (a-b)^2 \\ (a-b)(a+b) &= a^2-b^2 \end{aligned}} \\ &= \frac{(\sqrt{11})^2-2\sqrt{11}\times\sqrt{7}+(\sqrt{7})^2}{11-2} && \boxed{\because (a-b)^2 = a^2-2ab+b^2} \\ &= \frac{11-2\sqrt{11\times 7}+7}{4} \\ &= \frac{18-2\sqrt{77}}{4} = \frac{2(9-\sqrt{77})}{4} = \frac{9-\sqrt{77}}{2} = \frac{9}{2} - \frac{\sqrt{77}}{2} \end{aligned}$$

We have,

$$\begin{aligned} \frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} &= a-b\sqrt{77} \\ \Rightarrow \frac{9}{2} - \frac{\sqrt{77}}{2} &= a-b\sqrt{77} \\ \Rightarrow \frac{9}{2} - \frac{1}{2}\sqrt{77} &= a-b\sqrt{77} \end{aligned}$$

On equating the rational and irrational parts

We have

$$a = \frac{9}{2} \text{ and } b = \frac{1}{2}$$

(vi) Given that

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a+b\sqrt{5}$$

Rationalisation factor for $\frac{1}{a-b\sqrt{c}}$ is $a+b\sqrt{c} \Rightarrow$ for $\frac{1}{4-3\sqrt{5}}$ it is $4+3\sqrt{5}$

$$\begin{aligned} \Rightarrow \frac{4+3\sqrt{5}}{4-3\sqrt{5}} &= \frac{4+3\sqrt{5}}{4-3\sqrt{5}} \times \frac{4+3\sqrt{5}}{4+3\sqrt{5}} \\ &= \frac{(4+3\sqrt{5})^2}{4^2-(3\sqrt{5})^2} && \boxed{\begin{aligned} \because (a+b)(a+b) &= (a+b)^2 \\ (a-b)(a+b) &= a^2-b^2 \end{aligned}} \\ &= \frac{4^2 \times 2 \times 4 \times 3\sqrt{5} + (3\sqrt{5})^2}{16-3^2(\sqrt{5})^2} && \boxed{\because (a+b)^2 = a^2+2ab+b^2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{16+24\sqrt{5}+45}{16-45} = \frac{61+24\sqrt{5}}{-29} = \frac{-(61+24\sqrt{5})}{29} \\
 &= \frac{-61}{29} - \frac{24}{29}\sqrt{5}
 \end{aligned}$$

7. If $x = 2 + \sqrt{3}$, find the value of $x^3 + \frac{1}{x^3}$

Sol:

Given $x = 2 + \sqrt{3}$ and given to find the value of $x^3 + \frac{1}{x^3}$

We have $x = 2 + \sqrt{3}$

$$\Rightarrow \frac{1}{x} = \frac{1}{2 + \sqrt{3}}$$

Rationalization factor for $\frac{1}{a + \sqrt{b}}$ is $a - \sqrt{b}$

$$\begin{aligned}
 \Rightarrow \frac{1}{x} &= \frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} \\
 &= \frac{2 - \sqrt{3}}{2^2 - (\sqrt{3})^2} \quad \boxed{\because (a-b)(a+b) = a^2 - b^2} \\
 &= \frac{2 - \sqrt{3}}{4 - 3} = \frac{2 - \sqrt{3}}{1} = 2 - \sqrt{3} \\
 \therefore \frac{1}{x} &= 2 - \sqrt{3}
 \end{aligned}$$

And also, $\left(x + \frac{1}{x}\right) = 2 + \sqrt{3} + 2 - \sqrt{3} = 2 + 2 = 4$

$$\therefore \boxed{\left(x + \frac{1}{x}\right) = 4} \quad \dots\dots\dots(1)$$

We know that

$$\begin{aligned}
 x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right) \left(x^2 - x \times \frac{1}{x} + \frac{1}{x^2}\right) \\
 &= \left(x + \frac{1}{x}\right) \left(x^2 + \frac{1}{x^2} - 1\right) \\
 &= \left(x + \frac{1}{x}\right) \left(x^2 + \frac{1}{x^2} + 2 - 2 - 1\right) \\
 &= \left(x + \frac{1}{x}\right) \left(x^2 + \frac{1}{x^2} + 2 \cdot x \cdot \frac{1}{x} - 3\right)
 \end{aligned}$$

$$\begin{aligned} \Rightarrow x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right) \left(x^2 + \frac{1}{x^2} + 2 \cdot x \cdot \frac{1}{x} - 3\right) \\ &= \left(x + \frac{1}{x}\right) \left(\left(x + \frac{1}{x}\right)^2 - 3\right) \quad \left[\because (a+b)^2 = a^2 + 2ab + b^2\right] \end{aligned}$$

By putting $\left(x + \frac{1}{x}\right) = 4$, we get

$$\begin{aligned} x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right) \left[\left(x + \frac{1}{x}\right)^2 - 3\right] \\ &= (4)(4^2 - 3) \\ &= 4(16 - 3) \\ &= 4(13) \\ &= 52 \end{aligned}$$

\therefore The value of $x^3 + \frac{1}{x^3}$ is 52

8. If $x = 3 + \sqrt{8}$, find the value of $x^2 + \frac{1}{x^2}$

Sol:

$$\begin{aligned} \Rightarrow \frac{1}{x} &= \frac{1}{3 + \sqrt{8}} \times \frac{3 - \sqrt{8}}{3 - \sqrt{8}} \\ &= \frac{3 - \sqrt{8}}{3^2 - (\sqrt{8})^2} = \frac{3 - \sqrt{8}}{9 - 8} = \frac{3 - \sqrt{8}}{1} = 3 - \sqrt{8} \end{aligned}$$

$$\therefore \boxed{\frac{1}{x} = 3 - \sqrt{8}}$$

And also,

$$\left(x + \frac{1}{x}\right) = 3 + \sqrt{8} + 3 - \sqrt{8} = 3 + 3 = 6$$

$$\therefore \boxed{\left(x + \frac{1}{x}\right) = 6}$$

We know that

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

By putting $x + \frac{1}{x} = 6$ in the above

We get,

$$\begin{aligned}x^2 + \frac{1}{x^2} &= (6)^2 - 2 \\ &= 36 - 2 \\ &= 34\end{aligned}$$

∴ The value of $x^2 + \frac{1}{x^2}$ is 34.

Given that $x = 3 + \sqrt{8}$ and given to find the value of $x^2 + \frac{1}{x^2}$

We have $x = 3 + \sqrt{8}$

$$\Rightarrow \frac{1}{x} = \frac{1}{3 + \sqrt{8}}$$

The rationalization factor for $\frac{1}{a + \sqrt{b}}$ is $a - \sqrt{b}$

$$\Rightarrow \text{For } \frac{1}{3 + \sqrt{8}} \text{ is } 3 - \sqrt{8}$$

9. Find the value of $\frac{6}{\sqrt{5} - \sqrt{3}}$, it being given that $\sqrt{3} = 1.732$ and $\sqrt{5} = 2.236$.

Sol:

Given to find the value of $\frac{6}{\sqrt{5} - \sqrt{3}}$

Rationalisation factor for $\frac{1}{\sqrt{a} - \sqrt{b}}$ is $\sqrt{a} + \sqrt{b} \Rightarrow$ for $\frac{1}{\sqrt{5} - \sqrt{3}}$ is $\sqrt{5} + \sqrt{3}$

$$\begin{aligned}\Rightarrow \frac{6}{\sqrt{5} - \sqrt{3}} &= \frac{6}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} \\ &= \frac{6(\sqrt{5} + \sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} \quad [\because (a-b)(a+b) = a^2 - b^2] \\ &= \frac{6(\sqrt{5} + \sqrt{3})}{5 - 3} \\ &= \frac{6(\sqrt{5} + \sqrt{3})}{2} \\ &= 3(\sqrt{5} + \sqrt{3})\end{aligned}$$

We have $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$

$$\begin{aligned} \Rightarrow \frac{6}{5-\sqrt{3}} &= 3(2 \cdot 236 + 1 \cdot 732) \\ &= 3(3 \cdot 968) \\ &= 11.904 \\ \therefore \text{Value of } \frac{6}{5-\sqrt{3}} &\text{ is } 11.904 \end{aligned}$$

10. Find the values of each of the following correct to three places of decimals, it being given that $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.2360$, $\sqrt{6} = 2.4495$ and $\sqrt{10} = 3.162$

(i) $\frac{3-\sqrt{5}}{3+2\sqrt{5}}$

Sol:

(i) We have $\frac{3-\sqrt{5}}{3+2\sqrt{5}}$

Rationalization factor for $\frac{1}{a+b\sqrt{c}}$ is $a-b\sqrt{c} \Rightarrow$ for $\frac{1}{3+2\sqrt{5}}$ it is $3-2\sqrt{5}$

$$\begin{aligned} \Rightarrow \frac{3-\sqrt{5}}{3+2\sqrt{5}} &= \frac{3-\sqrt{5}}{3+2\sqrt{5}} \times \frac{3-2\sqrt{5}}{3-2\sqrt{5}} \\ &= \frac{3 \times 3 + 3 \times (-2\sqrt{5}) + (-\sqrt{5})(3) + (-\sqrt{5})(-2\sqrt{5})}{(3)^2 - (2\sqrt{5})^2} \end{aligned}$$

$$\therefore (a+b)(a-b) = a^2 - b^2$$

$$\begin{aligned} &= \frac{9 - 6\sqrt{5} - 3\sqrt{5} + 2 \times 5}{9 - 20} \\ &= \frac{9 + 10 - 9\sqrt{5}}{-11} = \frac{19 - 9\sqrt{5}}{-11} = \frac{9\sqrt{5} - 19}{11} \end{aligned}$$

We have $\sqrt{5} = 2.2360$

$$\begin{aligned} \Rightarrow \frac{3-\sqrt{5}}{3+2\sqrt{5}} &= \frac{9(2.2360) - 19}{11} \\ &= \frac{20.124 - 19}{11} \\ &= \frac{1.124}{11} \end{aligned}$$

$$= 0.102181818$$

≈ 0.102 (upto 3 decimals)

$$\therefore \text{The value of } \frac{3-\sqrt{5}}{3+2\sqrt{5}} = 0.102$$

11. If $x = \frac{\sqrt{3}+1}{2}$, find the value of $4x^3 + 2x^2 - 8x + 7$.

Sol:

Given $x = \frac{\sqrt{3}+1}{2}$ and given to find the value of $4x^3 + 2x^2 - 8x + 7$

$$\text{Now, } x = \frac{\sqrt{3}+1}{2}$$

$$\Rightarrow 2x = \sqrt{3}+1 \Rightarrow (2x-1) = \sqrt{3}$$

Squaring on both sides we get

$$(2x-1)^2 = (\sqrt{3})^2$$

$$\Rightarrow (2x)^2 - 2 \cdot 2x \cdot 1 + (1)^2 = 3$$

$$\therefore (a-b)^2 = a^2 - 2ab + b^2$$

$$\Rightarrow 4x^2 - 4x + 1 = 3$$

$$\Rightarrow 4x^2 - 4x + 1 - 3 = 0$$

$$\Rightarrow 4x^2 - 4x - 2 = 0$$

$$\Rightarrow 2(2x^2 - 2x - 1) = 0$$

$$\Rightarrow \boxed{2x^2 - 2x - 1 = 0}$$

Now take $4x^3 + 2x^2 - 8x + 7$

$$\Rightarrow 2x(2x^2 - 2x - 1) + 4x^2 + 2x + 2x^2 - 8x + 7$$

$$\Rightarrow 2x(2x^2 - 2x - 1) + 6x^2 - 6x + 7$$

$$\Rightarrow 2x(0) + 3(2x^2 - 2x - 1) + 7 + 3$$

$$\Rightarrow 0 + 3(0) + 10$$

$$\Rightarrow 10$$

\therefore The value of $4x^3 + 2x^2 - 8x + 7$ is 10.